Relations and Functions

Based on Types of Relations (Reflexive, Symmetric & Transitive)

1. The number of equivalence relations in the set $\{1,2,3\}$ containing the elements $(1,2)$ and $(2,1)$ is
(a) 0
(b) 1
(c) 2
(d) 3

[CBSE 2021-22]

Solution

(c): Equivalence relations in the set $\{1,2,3\}$ containing the elements $(1,2)$ and $(2,1)$ are
$R_1 = \{(1,2),(2,1),(1,1),(2,2),(3,3)\}$
$R_2 = \{(1,2),(2,1),(1,3),(3,1)(2,3),(3,2),(1,1),(2,2),(3,3)\}$
$\therefore$ Number of equivalence relations is 2

2. Let the relation R in the set $A = \{x \in Z: 0 \leq x \leq 12\}$, given by $R = \{(a,b): |a-b|$ is a multiple of 4 . $\}$ Then [1], the equivalence class containing 1 , is
(a) $\{1,5,9\}$
(b) $\{0,1,2,5\}$
(c) ϕ
(d) A

[CBSE SQP 2021-22]

Solution

(a): We have, $R = \{(a,b): |a-b|$ is a multiple of $4\}$
$\therefore$ The set of elements related to 1 is $\{1,5,9\}$.
So, equivalence class for [1] is $\{1,5,9\}$
So, equivalence class for [1] is $\{1,5,9\}$

3. Let R be the equivalence relation in the set $A = \{0,1,2,3,4,5\}$ given by $R = \{(a,b): 2$ divides $(a-b)\}$. Write the equivalence class [0].

[CBSE 2014]

Solution

Here, $R = \{(a,b) \in A \times A: 2$ divides $(a-b)\}$
This is the given equivalence relation, where $A = \{0,1,2,3,4,5\}$
$\therefore$ Equivalence class of [0] is $\{0,2,4\}$.

4. A relation R in $S = \{1,2,3\}$ is defined as $R = \{(1,1),(1,2),(2,2),(3,3)\}$. Which element(s) of relation R be removed to make R an equivalence relation?

[CBSE SQP 2021-22]

Solution

We have, $R = \{(1,1),(2,2),(3,3),(1,2)\}$
which is reflexive and transitive.
For R to be symmetric $(1,2)$ should be removed from R. (1)

5. Let R be the relation in the set Z of integers given by $R = \{(a,b): 2$ divides $a-b\}$. Show that the relation R is transitive? Write the equivalence class of 0

[CBSE SQP 2020-21]

Solution

Let 2 divides $(a-b)$ and 2 divides $(b-c)$: where $a,b,c \in Z$
So, 2 divides $[(a-b)+(b-c)]$
$\Rightarrow$ 2 divides $(a-c)$: Yes relation R is transitive.
Equivalence class of $0 = \{0, \pm 2, \pm 4, \pm 6, \ldots\}$

6. Let N be the set of natural numbers and R be the relation on $N \times N$ defined by $(a, b)R(c, d)$ iff $ad = bc$
for all $a, b, c, d \in N$. Show that R is an equivalence relation.

[CBSE 2020]

Solution

(i) Reflexivity : Let (a, b) be an arbitrary element of $N \times N$. Then, we have $(a, b) \in N \times N \forall a, b \in N$
$\Rightarrow ab = ba$
(By commutativity of multiplication)
$\Rightarrow (a, b)R(a, b)$
Thus, $(a, b)R(a, b)$ for all $(a, b) \in N \times N$
So, R is reflexive.
(ii) Symmetry: Let $(a, b), (c, d) \in N \times N$ such that $(a, b)R\ (c, d) \Rightarrow ad = bc \Rightarrow bc = ad \Rightarrow cb = da$
(By commutativity of multiplication) $\Rightarrow (c, d)R(a, b)$
Thus, $(a, b)R(c, d)$
$\Rightarrow (c, d)R(a, b)$ for all $(a, b), (c, d) \in N \times N$
So, R is symmetric.
(iii) Transitivity : Let $(a, b), (c, d), (e, f) \in N \times N$ such that $(a,$
b) $R(c, d)$ and $(c, d)R(e, f)$
Now, $(a, b)R(c, d) \Rightarrow ad = bc$
and $(c, d)R(e, f) \Rightarrow cf = de$
Multiplying (i) and (ii), we get $ad \cdot cf = bc \cdot de$
$\Rightarrow$ af $= be \Rightarrow (a, b)R(e, f)$
Thus, $(a, b)R(c, d)$ and $(c, d)R(e, f) \Rightarrow (a, b)R(e, f)$
So R is transitive.
$\therefore$ R is an equivalence relation.

7. Show that the relation R on $\mathbb{R}$ defined as $R = \{(a, b): a \leq b\}$, is reflexive and transitive but not symmetric.

[CBSE 2019]

Solution

We have, $R = \{(a, b): a \leq b, a, b \in \mathbb{R}\}$
(i) Reflexive: Since $a \leq a$
$\therefore$ $aRa \forall a \in R$
Hence, R is reflexive.
(ii) Symmetric: Let $a, b \in R$ such that

$$aRb \Rightarrow a \leq b \Rightarrow b \leq a$$

Hence R is not symmetric.
(iii) Transitive : Let $a, b, c \in R$
such that aRb and bRc
Now, $aRb \Rightarrow a \leq b$
... (i)
and $bRc \Rightarrow b \leq c$
From (i) and (ii), we have

$$a \leq b \leq c \Rightarrow a \leq c$$

$\therefore$ aRc
Hence, relation R is transitive.

8. Let $A = \{1,2,3, \ldots ,9\}$ and R be the relation in $A \times A$ defined by $(a,b)R(c,d)$ if $a + d = b + c$ for $(a,b),(c,d)$ in $A \times A$. Prove that R is an equivalence relation. Also obtain the equivalence class $[(2,5)]$.

[CBSE 2014]

Solution

Given $A = \{1,2,3,4, \ldots ,9\} \subset N$, the set of natural numbers.
To show: R is an equivalence relation.
(i) Reflexivity : Let (a,b) be an arbitrary element of $A \times A$. Then, we have $(a,b) \in A \times A \Rightarrow a,b \in A$
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$\Rightarrow a + b = b + a$ (by commutativity of addition on $\subset N$) $\Rightarrow (a,b)R(a,b)$
Thus, $(a,b)R(a,b)$ for all $(a,b) \in A \times A$
So, R is reflexive.
(ii) Symmetry: Let $(a,b),(c,d) \in A \times A$ such that $(a,b)R(c,d) \Rightarrow a + c'\ b + c \Rightarrow b + c = a + d$
$\Rightarrow c + b = d + a$
(by commutativity of addition on $\subset N$)
$\Rightarrow (c,d)R(a,b)$.
Thus, $(a,b)R(c,d)$
$\Rightarrow (c,d)R(a,b)$ for all $(a,b),(c,d) \in A \times A$.
So, R is symmetric.
(iii) Transitivity: Let $(a,b),(c,d),(e,f) \in A \times A$ such that $(a,b)R(c,d)$ and $(c,d)R(e,f)$
Now, $(a,b)R(c,d) \Rightarrow a + d = b + c$
and $(c,d)R(e,f) \Rightarrow c + f = d + e$
Adding (i) and (ii), we get
$(a + d) + (c + f) = (b + c) + (d + e)$
$\Rightarrow a + f = b + e \Rightarrow (a,b)R(e,f)$
(ii) Thus, $(a,b)R(c,d)$ and $(c,d)R(e,f) \Rightarrow (a,b)R(e,f)$.
So, R is transitive.
$\therefore$ R is an equivalence relation.
Equivalence class for $[(2,5)]$ is $\{(1,4),(2,5),(3,6),(4,7),(5,8),(6,9)\}$

9. Let N denote the set of all natural numbers and R be the relation on $N \times N$ defined by $(a,b)R(c,d)$ if $ad(b + c) = bc(a + d)$. Show that R is an equivalence relation

[CBSE 2015]

Solution

(i) Reflexivity : Let (a,b) be an arbitrary element of $N \times N$. Then, $(a,b) \in N \times N$
$\Rightarrow ab(b + a) = ba(a + b)$
[by commutativity of addition and multiplication on N] $\Rightarrow (a,b)R(a,b)$
So, R is reflexive on $N \times N$.
(ii) Symmetry : Let $(a,b),(c,d) \in N \times N$ be such that (a, b) $R(c,d)$.
$\Rightarrow ad(b + c) = bc(a + d) \Rightarrow cb(d + a) = da(c + b)$
[by commutativity of addition and multiplication on N] Thus, $(a,b)R(c,d) \Rightarrow (c,d)R(a,b)$ for all $(a,b),(c,d) \in N \times N$.
So, R is symmetric on N × N.
(iii) Transitivity: Let $(a,b),(c,d),(e,f) \in N \times N$ be such that $(a,b)R(c,d)$ and $(c,d)R(e,f)$. Then,
$(a,b)R(c,d) \Rightarrow ad(b + c) = bc(a + d)$
$\Rightarrow \dfrac{b + c}{bc} = \dfrac{a + d}{ad} \Rightarrow \dfrac{1}{b} + \dfrac{1}{c} = \dfrac{1}{a} + \dfrac{1}{d}$
and $(c,d)R(e,f) \Rightarrow cf(d + e) = \mathrm{de}(c + f)$

$\Rightarrow \frac{d+e}{de} = \frac{c+f}{cf} \Rightarrow \frac{1}{d} + \frac{1}{e} = \frac{1}{c} + \frac{1}{f}$

Adding (i) and (ii), we get

$\left(\frac{1}{b} + \frac{1}{c}\right) + \left(\frac{1}{d} + \frac{1}{e}\right) = \left(\frac{1}{a} + \frac{1}{d}\right) + \left(\frac{1}{c} + \frac{1}{f}\right)$

$\Rightarrow \frac{1}{b} + \frac{1}{e} = \frac{1}{a} + \frac{1}{f} \Rightarrow \frac{b+e}{be} = \frac{a+f}{af}$

$\Rightarrow af(b+e) = be(a+f)$

$\Rightarrow (a,b)R(e,f)$

So, R is transitive on $N \times N$.

Hence, R is an equivalence relation.

10. The function $f: N \to N$ is defined by

$$f(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$$

The function f is

(a) bijective (b) one-one but not onto

(c) onto but not one-one (d) neither one-one nor onto

[CBSE 2021-22]

Solution

(c): Given, $f(x) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$

Now, $f(1) = \frac{1+1}{2} = 1, f(2) = \frac{2}{2} = 1$

$\Rightarrow f(1) = f(2)$ but $1 \neq 2 \Rightarrow f$ is not one-one

But f is onto because range of f is N.

11. Show that the function $f: (-\infty, 0) \to (-1,0)$ defined by $f(x) = \frac{x}{1+|x|}, x \in (-\infty, 0)$ is one-one and onto.

[CBSE 2020]

Solution

Given, $f(x) = \frac{x}{1+|x|}, x \in (-\infty, 0)$

$= \frac{x}{1-x}$

$(\because x \in (-\infty, 0), |x| = -x)$

For one-one : Let $f(x_1) = f(x_2), x_1, x_2 \in (-\infty, 0)$

$\Rightarrow \frac{x_1}{1-x_1} = \frac{x_2}{1-x_2} \Rightarrow x_1(1-x_2) = x_2(1-x_1)$

$\Rightarrow x_1 - x_1x_2 = x_2 - x_1x_2 \Rightarrow x_1 = x_2$

Hence, if $f(x_1) = f(x_2)$, then $x_1 = x_2$

$\therefore$ f is one-one

For onto: Let $f(x) = y$

$$\Rightarrow y = \frac{x}{1-x} \Rightarrow y(1-x) = x \Rightarrow y - xy = x$$

$$\Rightarrow x + xy = y \Rightarrow x(1+y) = y \Rightarrow x = \frac{y}{1+y}$$

Here, $y \in (-1,0)$

So, x is defined for all values of y.

$\therefore$ f is onto

12. Let $A = \{3,5\}$. Then number of reflexive relations on A is

(a) 2

(b) 4

(c) 0

(d) 8

[CBSE 2023]

Solution

(b): $R_1 = \{(3,3),(5,5)\}$

$R_2 = \{(3,3),(5,5),(3,5)\} R_3 = \{(3,3),(5,5),(5,3)\} R_4 = \{(3,3),(5,5),(3,5),(5,3)\}$

13. Show that the relation S in set R of real numbers defined by

$$S = \{(a,b): a \leq b^3, a \in R, b \in R\}$$

is neither reflexive, nor symmetric, nor transitive.

[CBSE 2023]

Solution

Given, $S = \{(a,b): a \leq b^3, a \in R, b \in R\}$

For reflexive : For $(a,a) \in S, a \leq a^3$ which is not always true.

Let $a = \frac{1}{2}$ then $a^3 = \frac{1}{8} \Rightarrow a > a^3$

So, S is not reflexive.

For symmetric: For $(a,b) \in S \Rightarrow a \leq b^3$

Let $a = 1$ and $b = 2$

Then, $1 \leq 2^3$ i.e., $a \leq b^3$ but $2 \leq 1^3$ is not true $\Rightarrow (b,a) \notin S$

So, S is not symmetric.

For transitive : For $(a,b,c) \in S$

Consider $(a,b) \in S \Rightarrow a \leq b^3$

$$(b,c) \in S \Rightarrow b \leq c^3 \#(i)$$

From (i) and (ii)

$\Rightarrow a \leq c^9 \Rightarrow (a,c) \in S \ \therefore \ S$ is not transitive.

Hence, S is neither reflexive, nor symmetric nor transitive

14. Let R be the relation defined in the set $A = \{1,2,3,4, 5,6,7\}$ by $R = \{(a,b)$: both a and b are either odd or even]. Show that R is an equivalence relation. Hence, find the elements of equivalence class

[CBSE 2023]

Solution

Given, $A = \{1,2,3,4,5,6,7\}$ and

$R = \{(a,b)$: both a and b are either odd or even $\}$

$\Rightarrow R = [(1,1),(1,3),(1,5),(1,7),(3,3),(3,5),(3,7),(5,5),$
$(5,7),(7,7),(7,5),(7,3),(5,3),(7,1),(5,1),(3,1),(2,2),(2,4),$
$(2,6),(4,4),(4,6),(6,6),(6,4),(6,2),(4,2)\}$

For reflexivity: $(a,a) \in R$

Here, $(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7) \in R$

$\therefore$ R is reflexive.

For symmetric: If $(a,b) \in R$, then $(b,a) \in R$

Here, $(1,3),(3,1),(1,5),(5,1),(1,7),(7,1),(2,4),(4,2),(2,6),$
$(6,2),(3,5),(5,3),(3,7),(7,3),(4,6)(6,4),(5,7),(7,5) \in R$

$\therefore$ R is symmetric.

For transitivity: If $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$.

$\therefore$ If (a,b) are odd and (b,c) are odd, then (a,c) are also odd numbers.

$\therefore$ R is transitive.

Hence, R is an equivalence relation.

It is observed that all the element of subset (1,3,5,7) are odd. Thus, they are related to each other

because every element is odd.
Similarly, the element of subset {2,4,6} are even. Thus, they are related to each other because every element is even.
So, elements of equivalence class $[1] = \{1,3,5,7\}$

15. A relation R is defined on a set of real numbers $\mathbb{R}$ as $R = \{(x, y): x \cdot y$ is an irrational number $\}$. Check whether R is reflexive, symmetric and transitive or not.

[CBSE 2023]

Solution

` Relation R is defined on the set of real numbers as $R = \{(x, y): x \cdot y$ is an irrational number $\}$.
Reflexive : Let $x = \sqrt{3}$ which is an irrational number
Now $x \cdot x = x^2 = (\sqrt{3})^2 = 3$, which is not an irrational number. $(x, x) \notin R$
$\Rightarrow R$ is not reflexive.
Symmetric: Let $(x, y) \in R$, where $x \in R$ and $y \in R$ i.e., $x \cdot y$ is an irrational number.
As, we know that multiplication of two real numbers is commutative.
i.e., $x \cdot y = y \cdot x$
$\Rightarrow y \cdot x$ is also an irrational number.
$\Rightarrow (y, x) \in R$
$\therefore$ R is symmetric.
Transitive: Let $x = 2, y = \sqrt{3}$ and $z = 3$
Now, $x \cdot y = 2 \times \sqrt{3} = 2\sqrt{3}$, which is an irrational number.
$\therefore (x, y) \in R$
and $y \cdot z = \sqrt{3} \times 3 = 3\sqrt{3}$, which is an irrational number.
$\therefore (y, z) \in R$
But $x \cdot z = 2 \times 3 = 6$, which is not an irrational number.
$\Rightarrow (x, z) \notin R$
$\therefore$ R is not transitive.

16. If N denotes the set of all natural numbers and R is the relation on $N \times N$ defined by $(a, b)R(c, d)$, if $ad(b + c) = bc(a + d)$. Show that R is an equivalence relation.

[CBSE 2023]

Solution

Reflexive : Let (a, b) be an arbitrary element of N × N.
$\because$ $ab(b + a) = ba(a + b)$
[by commutativity of addition and multiplication on N]
$\Rightarrow (a, b)R(a, b)$
So, R is reflexive on $N \times N$.
Symmetric: Let $(a, b), (c, d) \in N \times N$ such that
$(a, b)R(c, d)$.
$\Rightarrow ad(b + c) = bc(a + d) \Rightarrow cb(d + a) = da(c + b)$
[by commutativity of addition and multiplication on N]
Thus, $(a, b)R(c, d) \Rightarrow (c, d)R(a, b)$ for all $(a, b), (c, d) \in N \times N$.
So, R is symmetric on $N \times N$.
Transitive: Let $(a, b), (c, d), (e, f) \in N \times N$ such that
$(a, b)R(c, d)$ and $(c, d)R(e, f)$. Then,
$(a, b)R(c, d) \Rightarrow ad(b + c) = bc(a + d)$

$$\Rightarrow \frac{b + c}{bc} = \frac{a + d}{ad} \Rightarrow \frac{1}{b} + \frac{1}{c} = \frac{1}{a} + \frac{1}{d} \#(i)$$

and $(c, d)R(e, f) \Rightarrow cf(d + e) = de(c + f)$

$$\Rightarrow \frac{d+e}{de} = \frac{c+f}{cf} \Rightarrow \frac{1}{d}+\frac{1}{e} = \frac{1}{c}+\frac{1}{f} \#(ii)$$

Adding (i) and (ii), we get

$\left(\frac{1}{b}+\frac{1}{c}\right)+\left(\frac{1}{d}+\frac{1}{e}\right) = \left(\frac{1}{a}+\frac{1}{d}\right)+\left(\frac{1}{c}+\frac{1}{f}\right) \Rightarrow \frac{1}{b}+\frac{1}{e} = \frac{1}{a}+\frac{1}{f} \Rightarrow \frac{b+e}{be} = \frac{a+f}{af} \Rightarrow af(b+e)$
$= be(a+f) \Rightarrow (a,b)R(e,f)$

So, R is transitive on $N \times N$.

Hence, R is an equivalence relation.

17. Case Study : An organization conducted bike race under two different categories - Boys and girls. There were 28 participants in all. Among all of them, finally three from category 1 and two from category 2 were selected for the final race. Ravi forms two sets B and G with these participants for his college project.

Let $B = \{b_1, b_2, b_3\}$ and $G = [g_1, g_2]$, where B represents the set of Boys selected and G the set of Girls selected for the final race.

Based on the above information, answer the following questions.

(i) How many relations are possible from B to ?

(ii) Among all the possible relations from B to G, how many functions can be formed from B to G ?

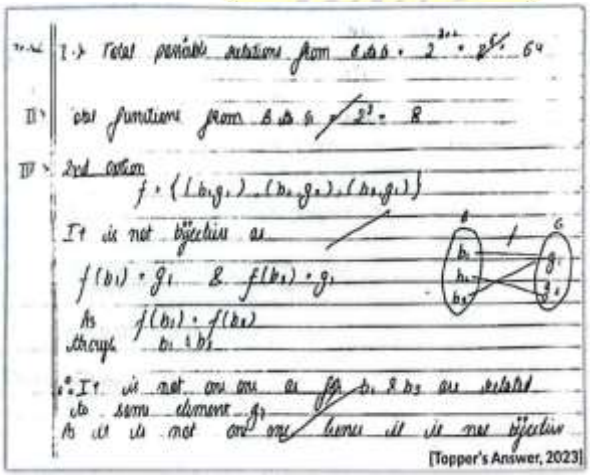
(iii) Let $R: B \to B$ be defined by $R = \{(x,y): x \text{ and } y \text{ are students of the same sex}\}$. Check if R is an equivalence relation.

OR

(iii) Afunction $f: B \to G$ be defined by $f = \{(b_1, g_1), (b_2, g_2), (b_3, g_1)\}$. Check if f is bijective, justify your answer.

[CBSE 2023]

Solution



OR

(iii) (First part)

We have, $R = [(x, y): x$ and y are students of the same sex]
Reflexive : Clearly R is reflexive, as $(x, x) \in R$
Symmetry: Let $(x, y) \in R$
$\Rightarrow$ x and y are of same sex
$\Rightarrow$ $(y, x) \in R$
$\therefore$ R is symmetric
Transitive : Let $(x, y) \in R$ and $(y, z) \in R$
$\Rightarrow$ x and y are of same sex and y and z are of same sex
$\Rightarrow$ x and z are of same sex
$\Rightarrow$ $(x, z) \in R$
$\therefore$ R is transitive.
Hence, R is an equivalence relation.

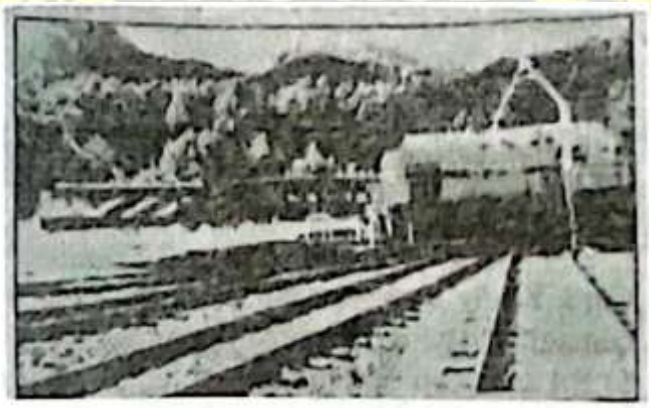
18. Assertion (A) : The relation $R = \{(x, y): (x + y)$ is a prime number and $x, y \in N\}$ is not a reflexive relation.
Reason (R) : The number ' $2n$ ' is composite for all natural numbers n.
(a) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A).
(b) Both Assertion (A) and Reason (R) are true but Reason (R) is not the correct explanation of Assertion (A).
(c) Assertion (A) is true but Reason (R) is false.
(d) Assertion (A) is false but Reason (R) is true.

[CBSE 2024]

Solution

(c): $\forall x \in N - \{1\}(x, x) \in R$ as $x + x = 2x$ is composite number i.e., not a prime number. So, R is not a reflexive relation.
Reason is false as for $n = 1, 2n$ is a prime number.
Hence, assertion (A) is true but reason (R) is false.

19. Case Study : (a) Students of a school are taken to a railway museum to learn about railways heritage and its history.



An exhibit in the museum depicted many rail lines on the track near the railway station. Let L be the set of all rail lines on the railway track and R be the relation on L defined by $R = \{(l_1, l_2): l_1$ is parallel to $l_2\}$.
On the basis of the above information, answer the following questions:
(i) Find whether the relation R is symmetric or not.
(ii) Find whether the relation R is transitive or not.
(iii) If one of the rail lines on the railway track is represented by the equation $y = 3x + 2$, then find the set of all rail lines in R related to it.

OR

(b) Let S be the relation defined by
$S = \{(I_1, I_2): I_1$ is perpendicular to $I_2\}$ check whether the relation S is symmetric and transitive.

[CBSE 2024]

Solution

Let $(l_1, l_2) \in R$
$\Rightarrow I_1$ is parallel to $I_2 \Rightarrow I_2$ is parallel to I_1
$\Rightarrow (I_2, I_1) \in R \Rightarrow R$ is symmetric.
(ii) Let $(I_1, I_2) \in R$ and $(I_2, I_3) \in R$
$\Rightarrow I_1$ is parallel to I_2 and I_2 is parallel to I_3
$\Rightarrow I_1$ is parallel to I_3
$\Rightarrow (I_1, I_3) \in R \Rightarrow R$ is transitive.
(iii) Given, line $y = 3x + 2$
Slope of line = 3
$\therefore$ The set of rail lines parallel to given line is y = 3x + c, where $c \in \mathbb{R}$.

OR

(b) Given, $S = \{(I_1, I_2): I_1$ is perpendicular to $I_2\}$
Let $(I_1, I_2) \in S$
$\Rightarrow I_1$ is perpendicular to $I_2 \Rightarrow I_2$ is perpendicular to I_1
$\Rightarrow (I_2, I_1) \in S \Rightarrow S$ is symmetric.
Let $(l_1, l_2) \in S$ and $(l_2, l_3) \in S$
$\Rightarrow I_1$ is perpendicular to I_2 and I_2 is perpendicular to I_3
$\Rightarrow I_1$ is parallel to I_3
$\Rightarrow (I_1, I_3) \notin S \Rightarrow S$ is not transitive.

20. Check whether the relation S in the set of real numbers R defined by $= \{(a, b)$: where $a - b + \sqrt{2}$ is an irrational number} is reflexive, symmetric or transitive.

[CBSE 2024]

Solution

. We have, $S = \{(a, b): a - b + \sqrt{2}$ is anirrational number $\}$
Reflexive :
$(a, a) \in S \Rightarrow a - a + \sqrt{2} = \sqrt{2}$ is always irrational.
$\therefore$ Sis reflexive.
Symmetric:
Let $(\sqrt{2}, 2) \in S$ as $\sqrt{2} - 2 + \sqrt{2}$
$= 2\sqrt{2} - 2$ is an irrational number.
but $(2, \sqrt{2}) \notin S$ as $2 - \sqrt{2} + \sqrt{2} = 2$, which is rational number.
$\therefore$ S is not symmetric.
Transitive:
Let $x = -\sqrt{2}, y = 3\sqrt{2}, z = 2$
$(-\sqrt{2}, 3\sqrt{2}) \in S$ as $-\sqrt{2} - 3\sqrt{2} + \sqrt{2} = -3\sqrt{2}$ is an irrational number.
$(3\sqrt{2}, 2) \in S$ as $3\sqrt{2} - 2 + \sqrt{2} = 4\sqrt{2} - 2$ is anirrational number.
But $(-\sqrt{2}, 2) \notin R$ as $-\sqrt{2} - 2 + \sqrt{2} = -2$ is a rational number.
$\therefore$ S is not transitive.

21. Let $f: R_+ \to [-5, \infty)$ be defined as $f(x) = 9x^2 + 6x - 5$, where $R_+$is the set of all non-negative real numbers. Then, f is :
(a) one-one
(b) onto

(c) bijective
(d) neither one-one nor onto

[CBSE 2024]

Solution

(c): Let $f(x_1) = f(x_2)$

$\Rightarrow 9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5 \Rightarrow 9(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0 \Rightarrow (x_1 - x_2)[9(x_1 + x_2) + 6] = 0$

$\Rightarrow x_1 - x_2 = 0$ as $9(x_1 + x_2) + 6 > 0$

($\because x_1, x_2$ are non-negative real numbers)

$\Rightarrow x_1 = x_2$

Thus, f is one-one.

Let $y \in [-5, \infty)$ be such that $f(x) = y$

Now, $f(x) = 9x^2 + 6x - 5 = 9x^2 + 6x + 1 - 6 = (3x + 1)^2 - 6$

$\Rightarrow y + 6 = (3x + 1)^2$

$\Rightarrow 3x + 1 = \sqrt{y + 6} \Rightarrow x = \frac{-1 + \sqrt{y+6}}{3}$

$\therefore f\left(\frac{-1+\sqrt{y+6}}{3}\right) = y$ $\therefore f(x)$ is onto.

So, the given function is bijective.

22. Show that a function $f: R \to R$ defined by $f(x) = \frac{2x}{1+x^2}$ is neither one-one nor onto. Further, find set A so that the given function $f: R \to A$ becomes an onto function.

[CBSE 2024]

Solution

We have, $f(x) = \frac{2x}{1+x^2}$

For one-one: Let $f(x_1) = f(x_2) \Rightarrow \frac{2x_1}{1+x_1^2} = \frac{2x_2}{1+x_2^2}$

$\Rightarrow x_1 + x_1x_2^2 = x_2 + x_1^2x_2 \Rightarrow (x_1 - x_2) + x_1x_2(x_2 - x_1) = 0$

$\Rightarrow (x_1 - x_2)(1 - x_1x_2) = 0 \Rightarrow x_1 = x_2$ or $x_1x_2 = 1$, if $x_1 \neq x_2$

$\Rightarrow f(x)$ is not one-one function.

For onto: Let $y = \frac{2x}{1+x^2} \Rightarrow y + x^2y - 2x = 0 \Rightarrow yx^2 - 2x + y = 0$

$\Rightarrow x = \frac{2 \pm \sqrt{4 - 4y^2}}{2y} \Rightarrow x = \frac{1 \pm \sqrt{1 - y^2}}{y}$

Now, for $f(x) = 2 = y$, there exists no real value of $x \in R$.

$\therefore f(x)$ is not onto.

We can see that, for $x \in R, \sqrt{1 - y^2} \geq 0$

$\Rightarrow 1 - y^2 \geq 0 \Rightarrow y^2 \leq 1 \Rightarrow -1 \leq y \leq 1$

So, $f(x)$ is onto if $y \in [-1,1]$

$\therefore$ Required set $A = [-1,1]$.

23. Assertion (A): Let $f(x) = e^x$ and $g(x) = \log x$. Then $(f + g)x = e^x + \log x$ where domain of $(f + g)$ is R. Reason (R): $\text{Dom}(f + g) = \text{Dom}(f) \cap \text{Dom}(g)$.

(a) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(b) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(c) Assertion (A) is true, but Reason (R) is false.

(d) Assertion (A) is false, but Reason (R) is true.

[CBSE 2025]

Solution

. (d): Since, domain of e^x is $(-\infty, \infty)$ and
Domain of $\log x$ is $(0, \infty)$
$\therefore$ Domain of $(f+g)x$ is the intersection of domain of $f(x)$ and $g(x)$, which is $(0, \infty)$
So, Assertion (A) is false.
$\therefore$ Assertion (A) is false, but Reason (R) is true.

24. Case Study : A school is organizing a debate competition with participants as speakers $S = \{S_1, S_2, S_3, S_4\}$ and these are judged by judges $J = \{J_1, J_2, J_3\}$. Each speaker can be assigned one judge. Let R be a relation from set S to J defined as $= \{(x, y)$: speaker x is judged by judge $y, x \in S, y \in J\}$.

Based on the above, answer the following :
(i) How many relations can be there from S to J ?
(ii) A student identifies a function from S to J as $f = \{(S_1, J_1), (S_2, J_2), (S_3, J_2), (S_4, J_3)\}$ Check if it is bijective.
(iii) (a) How many one-one functions can be there from set $\boldsymbol{S}$ to set $\boldsymbol{J}$?

OR

(iii) (b) Another student considers a relation $R_1 = \{(S_1, S_2), (S_2, S_4)\}$ in set S. Write minimum ordered pairs to be included in R_1 so that R_1 is reflexive but not symmetric.

[CBSE 2025]

Solution

. We have,
$S = [S_1, S_2, S_3, S_4], J = [J_1, J_2, J_3]$ and R be a relations defined as
$R = \{x, y\}$: speaker x is judged by judge $y, x \in S, y \in J)$
(i) Total possible relations from S to $J = 2^{4\times3} = 2^{12}$
(ii) Given, $f = \{(S_1, J_1), (S_2, J_2), (S_3, J_2), (S_4, J_3)\}$
Clearly, function is not one-one as S_2 and S_3 are mapping on J_2.
Therefore, f is not one-one.
Hence, it is not bijective.
(iii) (a) Number of one-one functions from S to $J = 0$ as $n < m$ OR
(iii) (b) Relation $R_1 = \{(S_1, S_2), (S_2, S_4)\}$ is defined on set S.
For R_1 to be reflexive, $(S_1, S_1), (S_2, S_2), (S_3, S_3), (S_4, S_4)$ must be added.
Hence, number of minimum ordered pairs to be included = 4

INVERSE TRIGONOMETRIC FUNCTIONS

1. $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$ is equal to

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) – 1 (d) 1

[CBSE SQP 2021-22]

Solution

(d) : We have,

$$\sin\left[\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right]$$

$$= \sin\left[\frac{\pi}{3} + \sin^{-1}\left(\frac{1}{2}\right)\right] = \sin\left[\frac{\pi}{3} + \frac{\pi}{6}\right] = \sin\left(\frac{\pi}{2}\right) = 1$$

2. Write the principal value of $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right)$.

[CBSE 2013]

Solution

$$\tan^{-1}(1) + \cos^{-1}\left(\frac{-1}{2}\right)$$

$$= \tan^{-1}\left(\tan\frac{\pi}{4}\right) + \cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right) = \frac{\pi}{4} + \frac{2\pi}{3} = \frac{11\pi}{12}$$

3. The value of $\sin^{-1}\left(\cos\frac{13\pi}{5}\right)$ is

(a) $-\frac{3\pi}{5}$ (b) $-\frac{\pi}{10}$ (c) $\frac{3\pi}{5}$ (d) $\frac{\pi}{10}$

[CBSE 2021-22]

Solution

(b) : We have, $\sin^{-1}\left(\cos\frac{13\pi}{5}\right) = \sin^{-1}\left[\cos\left(2\pi + \frac{3\pi}{5}\right)\right]$

$$= \sin^{-1}\left[\cos\frac{3\pi}{5}\right] = \sin^{-1}\left[\cos\left(\frac{\pi}{2} + \frac{\pi}{10}\right)\right]$$

$$= \sin^{-1}\left(-\sin\frac{\pi}{10}\right) = -\sin^{-1}\left(\sin\frac{\pi}{10}\right) = -\frac{\pi}{10}$$

4. Express $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right), \frac{-3\pi}{2} < x < \frac{\pi}{2}$ in the simplest form.

[CBSE SQP 2020-21]

Solution

We have,

$$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) = \tan^{-1}\left[\frac{\sin\left(\frac{\pi}{2}-x\right)}{1-\cos\left(\frac{\pi}{2}-x\right)}\right]$$

$$= \tan^{-1}\left[\frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right]$$

$$= \tan^{-1}\left[\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)\right] = \tan^{-1}\left[\tan\left\{\frac{\pi}{2}-\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right] = \frac{\pi}{4}+\frac{x}{2}$$

5. Simplest form of

$\tan^{-1}\left(\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right)$, $\pi < x < \frac{3\pi}{2}$ is

(a) $\frac{\pi}{4}-\frac{x}{2}$ (b) $\frac{3\pi}{2}-\frac{x}{2}$ (c) $-\frac{x}{2}$ (d) $\pi-\frac{x}{2}$

[CBSE SQP 2021-22]

Solution

(a) : We have,

$$\tan^{-1}\left(\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right), \pi < x < \frac{3\pi}{2}$$

$$= \tan^{-1}\left(\frac{\left|\sqrt{2}\cos\frac{x}{2}\right|+\left|\sqrt{2}\sin\frac{x}{2}\right|}{\left|\sqrt{2}\cos\frac{x}{2}\right|-\left|\sqrt{2}\sin\frac{x}{2}\right|}\right)$$

$$= \tan^{-1}\left(\frac{-\sqrt{2}\cos\frac{x}{2}+\sqrt{2}\sin\frac{x}{2}}{-\sqrt{2}\cos\frac{x}{2}-\sqrt{2}\sin\frac{x}{2}}\right) \left(\because \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}\right)$$

$$= \tan^{-1}\left(\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}\right) = \tan^{-1}\left(\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right) = \frac{\pi}{4}-\frac{x}{2}$$

Mixed Questions

6. Prove that:

$$\cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right)$$

[CBSE 2019]

Solution

Let $x = \cos^{-1}\left(\frac{12}{13}\right)$ and $y = \sin^{-1}\left(\frac{3}{5}\right)$

or $\cos x = \frac{12}{13}$ and $\sin y = \frac{3}{5}$

Now, $\sin x = \sqrt{1 - \cos^2 x}$ and $\cos y = \sqrt{1 - \sin^2 y}$

$$\Rightarrow \sin x = \sqrt{1 - \frac{144}{169}} \text{ and } \cos y = \sqrt{1 - \frac{9}{25}}$$

$$\Rightarrow \sin x = \frac{5}{13} \text{ and } \cos y = \frac{4}{5}$$

We know that,

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$= \frac{15}{13} \times \frac{4}{5} + \frac{12}{13} \times \frac{3}{5} = \frac{20}{65} + \frac{36}{65} = \frac{56}{65}$$

$$\Rightarrow x + y = \sin^{-1}\left(\frac{56}{65}\right)$$

$$or, \cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right)$$

7. Assertion (A) : The principal value of $\cot^{-1}(\sqrt{3})$ is $\frac{\pi}{6}$. Reason ($\mathbb{R}$) : Domain of $\cot^{-1} x$ is $\mathbb{R} - [-1,1]$.

(a) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(b) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(c) Assertion (A) is true and Reason (R) is false.

(d) Assertion (A) is false and Reason (R) is true.

[CBSE 2023]

Solution

(c): We know that, $\cot^{-1}(x), x \in (0, \pi)$

$\cot^{-1}(\sqrt{3}) = \cot^{-1}\left(\cot\frac{\pi}{6}\right) = \frac{\pi}{6}$ [$\because \cot^{-1}(\cot\theta) = \theta$]

$\therefore$ Assertion (A) is true.

Domain of $\cot^{-1} x$ is R.

So, reason (R) is false.

8. Express $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)$, where $\frac{-\pi}{2} < x < \frac{\pi}{2}$ in the simplest form.

OR

Simplify :

$$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)$$

[CBSE 2023]

Solution

. We have,

$$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)=\tan^{-1}\left[\frac{\sin\left(\frac{\pi}{2}-x\right)}{1-\cos\left(\frac{\pi}{2}-x\right)}\right]=\tan^{-1}\left[\frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right]$$

$$=\tan^{-1}\left[\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]=\tan^{-1}\left[\tan\left\{\frac{\pi}{2}-\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}\right]=\tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]=\frac{\pi}{4}+\frac{x}{2}$$

9. Evaluate : $\cos^{-1}\left[\cos\left(-\frac{7\pi}{3}\right)\right]$

[CBSE 2023]

Solution

$$\cos^{-1}\left[\cos\left(-\frac{7\pi}{3}\right)\right]=\cos^{-1}\left[\cos\left(\frac{7\pi}{3}\right)\right](\because \cos(-\theta)=\cos\theta)=\cos^{-1}\left(\cos\left(2\pi+\frac{\pi}{3}\right)\right)$$

$$=\cos^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right)$$

$$=\frac{\pi}{3}(\because \cos^{-1}(\cos x)=x\forall 0\le x\le\pi)$$

10. Find the domain of the function $f(x)=\sin^{-1}(x^2-4)$. Also, find its range.

[CBSE 2024]

Solution

For domain, $-1\le x^2-4\le 1$

$\Rightarrow 3\le x^2\le 5\Rightarrow\sqrt{3}\le|x|\le\sqrt{5}\Rightarrow x\in[-\sqrt{5},-\sqrt{3}]\cup[\sqrt{3},\sqrt{5}]$

Let $y=\sin^{-1}(x^2-4)\Rightarrow\sin y+4=x^2$

Now, $3\le x^2\le 5\Rightarrow 3\le\sin y+4\le 5\Rightarrow -1\le\sin y\le 1$

$$\Rightarrow -\frac{\pi}{2}\le y\le\frac{\pi}{2}\ \therefore R_f=\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$$

11. Find the domain of $f(x)=\cos^{-1}(1-x^2)$. Also, find its range.

[CBSE 2024]

Solution

. For domain, $-1\le 1-x^2\le 1,-2\le -x^2\le 0$

$\Rightarrow 0\le x^2\le 2\Rightarrow 0\le|x|\le\sqrt{2}\ \therefore D_f=[-\sqrt{2},\sqrt{2}]$

For range, let $y=\cos^{-1}(1-x^2)$

$\Rightarrow\cos y=1-x^2\Rightarrow x^2=1-\cos y$

But $0\le x^2\le 2\Rightarrow 0\le 1-\cos y\le 2$

$\Rightarrow -1\le-\cos y\le 1\Rightarrow -1\le\cos y\le 1$

$\Rightarrow 0\le y\le\pi$ [$\because \cos^{-1}x$ is decreasing function]

$\therefore\ R_f=[0,\pi]$

12. Simplify: $\cos^{-1}x+\cos^{-1}\left[\frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}\right];\frac{1}{2}\le x\le 1$

[CBSE 2024]

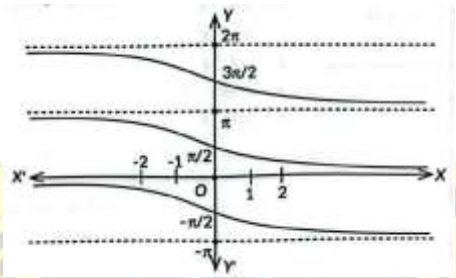
Solution

$$f(x) = \cos^{-1} x + \cos^{-1}\left[\frac{x}{2} + \frac{1}{2}\sqrt{3 - 3x^2}\right]$$
$$= \cos^{-1} x + \cos^{-1}\left[\frac{x}{2} + \frac{1}{2}\sqrt{3(1 - x^2)}\right]$$

Let $\cos\theta = x \Rightarrow \theta = \cos^{-1} x$

$$\therefore f(x) = \cos^{-1}(\cos\theta) + \cos^{-1}\left[\frac{\cos\theta}{2} + \frac{\sqrt{3}}{2}\sqrt{(1 - \cos^2\theta)}\right] \Rightarrow f(x) = \theta + \cos^{-1}\left[\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right]$$
$$\Rightarrow f(x) = \theta + \cos^{-1}\left[\cos\frac{\pi}{3}\cos\theta + \sin\frac{\pi}{3}\sin\theta\right] \Rightarrow f(x) = \theta + \cos^{-1}\left[\cos\left(\frac{\pi}{3} - \theta\right)\right]$$
$$\Rightarrow f(x) = \theta + \frac{\pi}{3} - \theta \Rightarrow f(x) = \frac{\pi}{3}$$

13. The graph shown below depicts:



(a) $y = \cot x$
(b) $y = \cot^{-1} x$
(c) $y = \tan x$
(d) $y = \tan^{-1} x$

[CBSE 2025]

Solution

(b)

14. If $y = \sin^{-1} x, -1 \le x \le 0$, then the range of y is

(a) $\left(\frac{-\pi}{2}, 0\right)$
(b) $\left[\frac{-\pi}{2}, 0\right]$
(c) $\left[\frac{-\pi}{2}, 0\right)$
(d) $\frac{-\pi}{2}, 0\Big]$

[CBSE 2025]

Solution

(b) : Given, $y = \sin^{-1} x, -1 \le x \le 0$
$\Rightarrow \sin^{-1}(-1) \le \sin^{-1} x \le \sin^{-1}(0)$
$\Rightarrow -\frac{\pi}{2} \le y \le 0 \ \therefore$ Range $= \left[-\frac{\pi}{2}, 0\right]$

15. Solve for x,

$$2\tan^{-1} x + \sin^{-1}\left(\frac{2x}{1 + x^2}\right) = 4\sqrt{3}$$

[CBSE 2025]

Solution

Given, $2\tan^{-1} x + \sin^{-1}\left(\frac{2x}{1+x^2}\right) = 4\sqrt{3}$

Putting $x = \tan\theta$ in $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, we get

$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1} x$

From (i), $2\tan^{-1} x + 2\tan^{-1} x = 4\sqrt{3}$

$\Rightarrow 4\tan^{-1} x = 4\sqrt{3} \Rightarrow \tan^{-1} x = \sqrt{3} \Rightarrow x = \tan\sqrt{3}$

16. Find the domain of $f(x) = \sin^{-1}(-x^2)$.

[CBSE 2025]

Solution

Given, $f(x) = \sin^{-1}(-x^2)$

Since the domain of $\sin^{-1} x$ is $[-1,1]$.

$\therefore -1 \leq -x^2 \leq 1 \Rightarrow -1 \leq x^2 \leq 1$

But $x^2 \geq 0$

$\therefore\ 0 \leq x^2 \leq 1 \Rightarrow -1 \leq x \leq 1$

$\therefore$ Domain of $f(x) = [-1,1]$

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Matrices

1. Write the number of all possible matrices of order 2 × 2 with each entry 1, 2 or 3.

[CBSE 2016]

Solution

As, matrix is of order 2 × 2, so there are 4 entries possible.

Each entry has 3 choices i.e., 1, 2 or 3. So, the number of ways make such matrices is 3 × 3 × 3 × 3 = 81.

2. The elements a_{ij} of 3 × 3 matrix are given by $a_{ij} = \frac{1}{2}|-3i + j|$. Write the value of elements a_{32}.

[CBSE 2014C]

Solution

Here, $a_{ij} = \frac{1}{2}|-3i + j|$

$\therefore \quad a_{32} = \frac{1}{2}|= 3.3 + 2| \qquad [For\ i = 3, j = 2]$

$= \frac{1}{2}|-9 + 2| = \frac{1}{2}|-7| = \frac{7}{2}$

3. If $\begin{bmatrix} 2a + b & a - 2b \\ 5c - d & 4c + 3d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}$, then value of $a + b - c + 2d$ is

(a) 8 (b) 10 (c) 4 (d) – 8

[CBSE SQP 2021-22]

Solution

(a) : From the definition of equality of two matrices, we have,

$2a + b = 4$ …(i)

$a - 2b = -3$ …(ii)

$5c - d = 11$ …(iii)

$4c + 3d = 24$ …(iv)

Solving (i) and (ii), we get

$5a = 5 \Rightarrow a = 1, b = 2$

Solving (iii) and (iv), we get

$19C = 57 \Rightarrow c = 3, d = 4$

$\therefore \quad a + b - c + 2d = 1 + 2 - 3 + 8 = 8$

4. If matrix A = $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $A^2 = kA$, then write the value of k.

[CBSE 2013]

Solution

Here, A = $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $A^2 = kA$

$\Rightarrow \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$\Rightarrow 2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \Rightarrow k = 2$

5. Given that $A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ and $A^2 = 3I$, then

(a) $1 + \alpha^2 + \beta\gamma = 0$ (b) $1 - \alpha^2 - \beta\gamma = 0$

(c) $3 - \alpha^2 - \beta\gamma = 0$ (d) $3 + \alpha^2 + \beta\gamma = 0$

[CBSE SQP 2021-22]

Solution

(c) : We have, $A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$

$\Rightarrow \quad A^2 = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix} = \begin{bmatrix} \alpha^2 + \beta\gamma & 0 \\ 0 & \gamma\beta + \alpha^2 \end{bmatrix}$

But $A^2 = 3I$

$\Rightarrow \quad \begin{bmatrix} \alpha^2 + \beta\gamma & 0 \\ 0 & \alpha^2 + \beta\gamma \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$\Rightarrow \quad \alpha^2 + \beta\gamma = 3 \Rightarrow 3 - \alpha^2 - \beta\gamma = 0$

6. Find matrix A such that

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$$

[CBSE 2017]

Solution

Given that, $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$

Let X = $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}_{3\times2}$ $and\ Y = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}_{3\times2}$

As order of X is 3 × 2, then A should be of order 2 × 2, so that we get Y matrix of order 3 × 2.

Let A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Now, $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} 2a-c & 2b-d \\ a+0 & b+0 \\ -3a+4c & -3b+4d \end{bmatrix} = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$

On comparing the corresponding elements, we get

$2a - c = -1$ …(i)

$2b - d = -8$ …(ii)

$a = 1$ …(iii)

and $b = -2$ …(iv)

Substituting $a = 1$ in (i), we get

$c = 3$

and substituting $b = -2$ in (ii), we get

$d = 4$

So, A = $\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$

7. If A = $\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, find $A^2 - 5A + 4I$ and hence find a matrix X such that $A^2 - 5A + 4I + X = O$.

[CBSE 2015]

Solution

Given, A = $\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

Now, $A^2 - 5A + 4I$

$= \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -1 & 2 \\ 9 & 2 & 5 \\ 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -3 \\ -1 & -3 & -10 \\ -5 & 4 & 2 \end{bmatrix}$$

Since, $A^2 - 5A + 4I + X = O$

$\Rightarrow X = -(A^2 - 5A + 4I)$

$$\therefore \quad X = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 10 \\ 5 & -4 & -2 \end{bmatrix}$$

8. If $[2x \quad 3]\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ 3 \end{bmatrix} = O$, find *x*.

[CBSE 2015C]

Solution

Here, $[2x \quad 3]\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ 3 \end{bmatrix} = 0$

$$\Rightarrow \quad [2x \quad 3]\begin{bmatrix} x+6 \\ -3x \end{bmatrix} = 0$$

$$\Rightarrow \quad 2x(x+6) + 3(-3x) = 0 \Rightarrow 2x^2 + 12x - 9x = 0$$

$$\Rightarrow \quad 2x^2 + 3x = 0 \Rightarrow x(2x+3) = 0 \Rightarrow x = 0, \frac{-3}{2}.$$

9. If A is a square matrix such that $A^2 = A$, then $(I - A)^3 + A$ is equal to

(a) I (b) O (c) I – A (d) I + A

[CBSE 2020]

Solution

(a) : We have, $A^2 = A$

Now, $(I - A)^3 + A = (I - A)(I - A)(I - A) + A$

$= (I.I - I.A - A.I + A.A)(I - A) + A$

$= (I - A - A + A)(I - A) + A$ $\quad [\because \boldsymbol{I.A = A.I = A \text{ and } A^2 = A}]$

$= (I - A)(I - A) + A$

$= (I.I - I.A - A.I + A.A) + A$

$= (I - A - A + A) + A = (I - A) + A = I$

OR

$(I - A)^3 + A$

$= I^3 - A^3 + 3A^2I - 3A + A$

$= I - A^2A + 3A^2 - 3A + A$

$= I - A^2 + 3A - 3A + A$

$= I - A + 3A - 3A + A$

= I

A . I

10. Three schools A, B and C organized a mela for collecting funds for helping the rehabilitation of flood victims. They sold hand made fans, mats and plates from recycled material at a cost of Rs. 25, Rs. 100 and Rs. 50 each. The number of articles sold are given below.

Article/School	A	B	C
Hand-fans	40	25	35
Mats	50	40	50
Plates	20	30	40

Find the funds collected by each school separately by selling the above articles. Also, find the total funds collected for the purpose.

Write one value generated by the above situation.

[CBSE 2015]

Solution

The number of articles sold by each school can be written in the matrix form as

$$X = \begin{bmatrix} 40 & 25 & 35 \\ 50 & 40 & 50 \\ 20 & 30 & 40 \end{bmatrix}$$

The cost of each article can be written in the matrix form as $Y = \begin{bmatrix} 25 & 100 & 50 \end{bmatrix}$

The fund collected by each school is given by

$$YX = \begin{bmatrix} 25 & 100 & 50 \end{bmatrix} \begin{bmatrix} 40 & 25 & 35 \\ 50 & 40 & 50 \\ 20 & 30 & 40 \end{bmatrix} = \begin{bmatrix} 7000 & 6125 & 7875 \end{bmatrix}$$

Therefore, the funds collected by schools A, B and C are Rs. 7000, Rs. 6125 and Rs. 7875 respectively.

Thus, the total funds collected

= Rs. (7000 + 6125 + 7875) = Rs. 21000

The situation highlights the helping nature of the students.

11. To promote the making of toilets for women, an organisation tried to generate awareness through (i) house calls (ii) letters and (iii) announcements. The cost for each mode per attempt is given below:

(i) Rs. 50 (ii) Rs. 20 (iii) Rs. 40

The number of attempts made in three villages X, Y and Z are given below:

	(i)	(ii)	(iii)
X	400	300	100
Y	300	250	75
Z	500	400	150

Find the total cost incurred by the organisation for the three villages separately, using matrices. Write one value generated by the organisation in the society.

[CBSE 2015]

Solution

Let Rs. A, Rs. B and Rs. C be the cost incurred by the organization for villages X, Y and Z respectively. Then, we get the matrix equation as

$$\begin{bmatrix}400 & 300 & 100\\ 300 & 250 & 75\\ 500 & 400 & 150\end{bmatrix}\begin{bmatrix}50\\20\\40\end{bmatrix}=\begin{bmatrix}A\\B\\C\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}A\\B\\C\end{bmatrix}=\begin{bmatrix}400\times 50+300\times 20+100\times 40\\ 300\times 50+250\times 20+75\times 40\\ 500\times 50+400\times 20+150\times 40\end{bmatrix}$$

$$=\begin{bmatrix}20{,}000+6{,}000+4{,}000\\ 15{,}000+5{,}000+3{,}000\\ 25{,}000+8{,}000+6{,}000\end{bmatrix}=\begin{bmatrix}30{,}000\\23{,}000\\39{,}000\end{bmatrix}$$

$$\therefore \quad A=\ Rs.30{,}000, B=Rs.23{,}000\ and\ C=Rs.39{,}000$$

These are the costs incurred by the orgainisation for villages X, Y and Z respectively.

The value generated by the organization in the society is cleanliness.

12. In a parliament election, a political party hired a public relations firm to promote its candidates in three ways-telephone, house calls and letters. The cost per contact (in paise) is given in matrix A as

$$A=\begin{bmatrix}140\\200\\150\end{bmatrix}\begin{matrix}\text{Telephone}\\ \text{House call}\\ \text{Letters}\end{matrix}$$

The number of contacts of each type made in two cities X and Y is given in matrix B as

$$B = \begin{matrix} \text{Telephone} & \text{House call} & \text{Letters} \\ \end{matrix}$$
$$\begin{bmatrix} 1000 & 500 & 5000 \\ 3000 & 1000 & 10000 \end{bmatrix} \begin{matrix} \text{City X} \\ \text{City Y} \end{matrix}$$

Find the total amount spent by the party in the two cities. What should one consider before casting his/ her vote-party's promotional activity or their social activities? [CBSE 2015]

Solution

The total amount spent by the party in two cities X and Y is represented in the matrix equation by matrix C as,

$C = BA$

$$\Rightarrow \quad \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1000 & 500 & 5000 \\ 3000 & 1000 & 10000 \end{bmatrix} \begin{bmatrix} 140 \\ 200 \\ 150 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1000 \times 140 + 500 \times 200 + 5000 \times 150 \\ 3000 \times 140 + 1000 \times 200 + 10000 \times 150 \end{bmatrix}$$

$$= \begin{bmatrix} 990000 \\ 2120000 \end{bmatrix}$$

$\Rightarrow \quad X = 990000$ paise, $Y = 2120000$ paise

$\therefore \quad X =$ Rs. 9900 and $Y =$ Rs. 21200

i.e., Amount spent by the part in city X and Y is Rs. 9900 and Rs. 21200 respectivley. One should consider about the social activities of a political party before casting his/her vote.

13. A trust fund, Rs. 35,000 is to be invested in two different types of bonds. The first bond pays 8% interest per annum which will be given to orphanage and second bond pays 10% interest per annum which will be given to an N.G.O. (Cancer Aid Society). Using matrix multiplication, determine how to divide Rs. 35,000 among two types of bonds if the trust fund obtains an annual total interest of Rs. 3,200. What are the values reflected in this question?

[CBSE 2015C]

Solution

Trust fund = Rs. 35,000.

Let Rs. X be invested in the first bond and then Rs. (35,000 – x) will be invested in the second bond.

Interest paid on the first bond = 8% = 0.08

Interest paid on the second bond = 10% = 0.10

Total annual interest = Rs. 3200.

$\therefore$ In matrices, $[x \quad 35{,}000 \quad -x]\begin{bmatrix}0.08\\1.10\end{bmatrix} = [3{,}200]$

$\Rightarrow \quad x \times 0.08 + (35{,}000 - x) \times 0.10 = 3{,}200$

$\Rightarrow \quad x \times \frac{8}{100} + (35{,}000 - x) \times \frac{10}{100} = 3{,}200$

$\Rightarrow \quad 8x + 3{,}500 - 10x = 3{,}20{,}000$

$\Rightarrow \quad 2x = 30{,}000 \Rightarrow x = 15{,}000$

$\therefore$ Rs. 15,000 should be invested in the first bond and Rs. 35,000 – Rs. 15,000 = Rs. 20,000 should be invested in the second bond.

14. The values reflected in this question are:

(i) Spirit of investemnet.

(ii) Giving charity to cancer patients.

(iii) Helping the orphans living in the society.

(iv) If the matric A $= \begin{bmatrix}0 & a & -3\\2 & 0 & -1\\b & 1 & 0\end{bmatrix}$ is skew symmetric, find the value of $'a'$ and $'b'$.

[CBSE 2018]

Solutio

A square matrix A is said to be skew symmetric matrix if A = – A' …(i)

Now, A $= \begin{bmatrix}0 & a & -3\\2 & 0 & -1\\b & 1 & 0\end{bmatrix}$ $\therefore$ $A' = \begin{bmatrix}0 & 2 & b\\a & 0 & 1\\-3 & -1 & 0\end{bmatrix}$

From (i), A + A' = 0

$\Rightarrow \begin{bmatrix}0 & a & -3\\2 & 0 & -1\\b & 1 & 0\end{bmatrix} + \begin{bmatrix}0 & 2 & b\\a & 0 & 1\\-3 & -1 & 0\end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix}0 & 2+a & b-3\\a+2 & 0 & 0\\b-3 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$

$\Rightarrow \quad a = 2 = 0 \;\&\; b - 3 = 0$

$\therefore \quad a = -2 \;\&\; b = 3$

15. If A and B are symmetric matrices, such that AB and BA are both defined, then prove that AB – BA is a skew symmetric matrix.

[CBSE 2019]

Solution

Given, A and B are symmetric matrices

$\therefore \quad A' = A \text{ and } B' = B$

Now, $(AB - BA)' - (BA)'$

$= (B'A') - (A'B')$

$= (BA - AB) \qquad [\because A' = A \text{ and } B' = B]$

$= -(AB - BA)$

Thus, $(AB - BA)' = -(AB - BA)$

Hence, (AB – BA) is a skew symmetric matrix.

16. Show that all the diagonal elements of a skew symmetric matrix are zero.

[CBSE 2017]

Solution

Let A = $[a_{ij}]$ be a skew symmetric matrix

Then, $a_{ji} = -a_{ij} \; \forall \; i, j$

$\Rightarrow \quad a_{ij} = -a_{ij} \; \forall \; i \; \Rightarrow 2a_{ij} = 0 \; \Rightarrow a_{ij} = 0 \; \forall \; j$

$\Rightarrow \quad a_{11} = a_{22} = a_{33} = \cdots \ldots = a_{nn} = 0$

17. Express the matric A = $\begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

[CBSE 2015C]

Solution

We know that a square matrix A can be written as

$\boldsymbol{A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)}$

Out of which $\frac{1}{2}(A + A^T)$ is symmetric and $\frac{1}{2}\ (A - A^T)$ is skew symmetric matrix.

$\therefore \quad$ For the given matrix

A = $\begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix}, A^T = \begin{bmatrix} 2 & 7 & 1 \\ 4 & 3 & -2 \\ -6 & 5 & 4 \end{bmatrix}$

$\therefore \quad A + A^T = \begin{bmatrix} 4 & 11 & -5 \\ 11 & 6 & 3 \\ -5 & 3 & 8 \end{bmatrix}$ and

$A - A^T = \begin{bmatrix} 0 & -3 & -7 \\ 3 & 0 & 7 \\ 7 & -7 & 0 \end{bmatrix}$

Hence, A = $\frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

$= \begin{bmatrix} 2 & 11/2 & -5/2 \\ 11/2 & 3 & 3/2 \\ -5/2 & 3/2 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -3/2 & -7/2 \\ 3/2 & 0 & 7/2 \\ 7/2 & -7/2 & 0 \end{bmatrix}$

In which first is symmetric and the second is skew symmetric matrix.

18. If A = $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$. Hence find A^{-1}.

[CBSE SQP 2020-21]

Solution

19. If matrix $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $A^2 = kA$, then the value of k is :

(a) 1
(b) -2
(c) 2
(d) -1

[CBSE 2023]

Solution

(c) : Since, $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$\Rightarrow A^2 = A \cdot A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 2A$

On comparing, $k = 2$

20. It is given that $X\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$. Then matrix X is :

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
(b) $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$
(c) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
(d) $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

[CBSE 2023]

Solution

(c) : Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

We have, $X\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 3a + b & 2a - b \\ 3c + d & 2c - d \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

On comparing the element of matrices, we get

$$3a + b = 4 \#(i)$$
$$2a - b = 1 \#(ii)$$
$$3c + d = 2 \#(iii)$$
$$2c - d = 3 \#(iv)$$

Adding (i) and (ii), we get $5a = 5 \Rightarrow a = 1$

Putting $a = 1$ in (i), we get $3(1) + b = 4 \Rightarrow b = 1$

Adding (iii) and (iv), we get $5c = 5 \Rightarrow c = 1$

Putting $c = 1$ in (iii), we get $3 + d = 2 \Rightarrow d = -1$

$$\therefore \text{X} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

21. If $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $(3I + 4A)(3I - 4A) = x^2 I$, then the value(s) x is/are :

(a) $\pm\sqrt{7}$

(b) 0

(c) ± 5

(d) 25

[CBSE 2023]

Solution

. (c) : Given, $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Now, $3I + 4A = 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 4\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$

and $3I - 4A = 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 4\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$

Consider, $(3I + 4A) \cdot (3I - 4A) = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$

$= \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} = 25\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 25I$

As, $(3I + 4A)(3I - 4A) = x^2 I$

$$\Rightarrow 25I = x^2 I \Rightarrow x^2 = 25 \Rightarrow x = \pm 5$$

22. If A is a square matrix and $A^2 = A$, then $(I + A)^2 - 3A$ is equal to

(a) 1

(b) A

(c) $2A$

(d) 31

[CBSE 2023]

Solution

(a): Given that $A^2 = A$

Consider $(I + A)^2 - 3A = I^2 + A^2 + 2AI - 3A$

$= I + A + 2A - 3A$ [$\because I^2 = I, A^2 = A$ (given)]

$= I$

23. If for a square matrix $A, A^2 - 3A + I = O$ and $A^{-1} = xA + yI$, then the value of $x + y$ is :

(a) -2

(b) 2

(c) 3

(d) -3

[CBSE 2023]

Solution

(b) : Given, $A^2 - 3A + 1 = 0$

and $A^{-1} = xA + yI$

where A is a square matrix

Now, pre-multiply (i) by A^{-1} on both sides, we get

$A^{-1}A^2 - 3A^{-1}A + A^{-1} = A^{-1}O \Rightarrow A - 3I + A^{-1} = 0 \Rightarrow A^{-1} = -A + 3I$

On comparing with equation (ii), we get

$x = -1$ and $y = 3$

$$\therefore\ x + y = -1 + 3 = 2$$

24. If $\begin{bmatrix} a & c & 0 \\ b & d & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a scalar matrix, then the value of $a + 2b + 3c + 4d$ is :

(a) 0

(b) 5

(c) 10

(d) 25

[CBSE 2024]

Solution

(d) : We have, $\begin{bmatrix} a & c & 0 \\ b & d & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a scalar matrix.

$\therefore\ a = d = 5, b = c = 0$

$a + 2b + 3c + 4d = 5 + 2 \times 0 + 3 \times 0 + 4 \times 5 = 5 + 20 = 25$

25. If the sum of all elements of a 3×3 scalar matrix is 9, then the product of all its elements is :

(a) 0

(b) 9

(c) 27

(d) 729

[CBSE 2024]

Solution

(a) : As we know that in a scalar matrix, every nondiagonal element is zero.

$\therefore$ Product of all elements of the given scalar matrix $= 0$.

26. If $A = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$, then the value of $I - A + A^2 - A^3 + \cdots$ is :

(a) $\begin{bmatrix} -1 & -1 \\ 4 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

[CBSE 2024]

Solution

. (a) : Given $A = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$, then

$A^2 = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \Rightarrow A^n = 0 \forall n \geq 2 \therefore I - A + A^2 - A^3 + \cdots = I - A$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 4 & 3 \end{bmatrix}$

27. Find the matrix A^2, where $A = [a_{ij}]$ is a 2×2 matrix whose elements are given by a_{ij} = maximum (i, j) minimum (i, j) :

(a) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

[CBSE 2024]

Solution

(c) : Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$a_{11} = \max(1,1) - \min(1,1) = 1 - 1 = 0 a_{12} = \max(1,2) - \min(1,2) = 2 - 1 = 1 a_{21}$

$= \max(2,1) - \min(2,1) = 2 - 1 = 1 a_{22} = \max(2,2) - \min(2,2) = 2 - 2 = 0 \therefore A$

$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

28. If A and B are square matrices of order m such that $A^2 - B^2 = (A - B)(A + B)$, then which of the following is always correct ?

(a) $A = B$

(b) $AB = BA$

(c) $A = 0$ or $B = 0$

(d) $A = 1$ or $B = 1$

[CBSE 2025]

Solution

b) : We have, $A^2 - B^2 = (A - B)(A + B)$

R.H.S. $= (A - B)(A + B) = A(A + B) - B(A + B)$

$= A^2 + AB - BA - B^2$

It only satisfies the condition if $AB = BA$

29. Let $A = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 4 & 2 \\ 12 & 16 & 8 \\ -6 & -8 & -4 \end{bmatrix}$ be two matrices. Then, find the matrix B if $AB = C$.

[CBSE 2025]

Solution

. Since, A is 3×1 matrix. So, B must be 1×3 matrix.

Let $B = [x \quad y \quad z]$

Given, $AB = C$

$$\Rightarrow \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} [x \quad y \quad z] = \begin{bmatrix} 3 & 4 & 2 \\ 12 & 16 & 8 \\ -6 & -8 & -4 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} x & y & z \\ 4x & 4y & 4z \\ -2x & -2y & -2z \end{bmatrix} = \begin{bmatrix} 3 & 4 & 2 \\ 12 & 16 & 8 \\ -6 & -8 & -4 \end{bmatrix}$$

On comparing the elements on both sides, we get

$$= 3, y = 4, z = 2$$

$$\therefore \ B = [3 \quad 4 \quad 2]$$

30. Let A be a matrix of order $m \times n$ and B is a matrix such that $A^T B$ and BA^T are defined. Then, the order of B is :

(a) $m \times m$

(b) $n \times n$

(c) $m \times n$

(d) $n \times m$

[CBSE 2025]

Solution

(c): We have, A is a matrix of order $m \times n$ and $A^\top B$ and BA^T are defined, then order of B is $m \times n$.

31. If $A = \begin{bmatrix} 1 & 12 & 4y \\ 6x & 5 & 2x \\ 8x & 4 & 6 \end{bmatrix}$ is a symmetric matrix, then $(2x + y)$ is

(a) -8

(b) 0

(c) 6

(d) 8

[CBSE 2025]

Solution

. (d) : Given, A is symmetric matrix, then $A' = A$

$$A = \begin{bmatrix} 1 & 12 & 4y \\ 6x & 5 & 2x \\ 8x & 4 & 6 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 1 & 6x & 8x \\ 12 & 5 & 4 \\ 4y & 2x & 6 \end{bmatrix}$$

Now, $A' = A$

$$\Rightarrow \begin{bmatrix} 1 & 6x & 8x \\ 12 & 5 & 4 \\ 4y & 2x & 6 \end{bmatrix} = \begin{bmatrix} 1 & 12 & 4y \\ 6x & 5 & 2x \\ 8x & 4 & 6 \end{bmatrix}$$

By equality of two matrices, we get

$$6x = 12 \Rightarrow x = 2 \text{ and } 8x = 4y$$

$$\Rightarrow y = 2x = 2(2) = 4$$

$$\therefore \ 2x + y = 2(2) + 4 =$$

32. If $A = \begin{bmatrix} 0 & -3 & 8 \\ 3 & 0 & 5 \\ -8 & -5 & 0 \end{bmatrix}$, then A is a :

(a) null matrix

(b) symmetric matrix

(c) skew-symmetric matrix

(d) diagonal matrix

[CBSE 2025]

Solution

(c) : Given $A = \begin{bmatrix} 0 & -3 & 8 \\ 3 & 0 & 5 \\ -8 & -5 & 0 \end{bmatrix}$

$$\Rightarrow A' = \begin{bmatrix} 0 & -3 & 8 \\ 3 & 0 & 5 \\ -8 & -5 & 0 \end{bmatrix}'$$

$$= \begin{bmatrix} 0 & 3 & -8 \\ -3 & 0 & -5 \\ 8 & 5 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -3 & 8 \\ 3 & 0 & 5 \\ -8 & -5 & 0 \end{bmatrix} = -A$$

Hence, A is skew-symmetric matrix.

33. Which of the following can be both a symmetric and skew-symmetric matrix?

(a) Unit matrix

(b) Diagonal matrix

(c) Null matrix

(d) Rowmatrix

[CBSE 2025]

Solution

(c) : Null matrix is the only matrix that can be both symmetric and skew-symmetric

Determinant

Based on Determinant

1. Value of k, for which A = $\begin{bmatrix} k & 8 \\ 4 & 2k \end{bmatrix}$ is a singular matrix is

(a) 4 (b) – 4 (c) ± 4 (d) 0

[CBSE SQP 2021-22]

Solutions

(c) : $\because$ A is singular matrix.

$\therefore \quad |A| = 0$

$\Rightarrow \quad \begin{vmatrix} k & 8 \\ 4 & 2k \end{vmatrix} = 0 \Rightarrow 2k^2 - 32 = 0 \Rightarrow k^2 = 16 \Rightarrow k = \pm 2\ 4$

2. Value of k, for which A = $\begin{bmatrix} k & 8 \\ 4 & 2k \end{bmatrix}$ is a singular matrix is

(a) 4 (b) – 4 (c) ± 4 (d) 0

[CBSE SQP 2021-22]

Solutions

(c) : $\because$ A is singular matrix.

$\therefore \quad |A| = 0$

$\Rightarrow \quad \begin{vmatrix} k & 8 \\ 4 & 2k \end{vmatrix} = 0 \Rightarrow 2k^2 - 32 = 0 \Rightarrow k^2 = 16 \Rightarrow k = \pm 2\ 4$

3. If $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix} = 8$, write the value of x.

[CBSE 2016]

Solutions

Given, $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix} = 8$

$\Rightarrow \quad x(-x^2 - 1) - \sin\theta\,(-x\sin\theta - \cos\theta) + \cos\theta\ (-\sin\theta + x\cos\theta) = 8$

$\Rightarrow \quad -x^3 - x + x\sin^2\theta + \sin\theta\cos\theta - \sin\theta\cos\theta + x\cos^2\theta = 8$

$\Rightarrow \quad -x^3 - x + x(\sin^2\theta + \cos^2\theta) = 8$

$\Rightarrow \quad -x^3 - x + x = 8 \Rightarrow x^3 + 8 = O \qquad [\because x^2 - 2x + 4 > O\ \forall\, x]$

$\Rightarrow \quad x = -2$

4. If A = $\begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = 27$, then the value of α is

(a) ± 1 (b) ± 2 (c) $\pm \sqrt{5}$ (d) $\pm \sqrt{7}$

[CBSE 2021-22]

Solutions

(d): Given, A = $\begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ …(i)

$|A^3| = 27$

$\Rightarrow \quad |A|^3 = 27 \quad [\because |\boldsymbol{A^n}| = |\boldsymbol{A}|^{\boldsymbol{n}}] \Rightarrow A = 3$ ...(ii)

From (i) and (ii), we get

$\Rightarrow \quad \alpha^2 - 4 = 3 \Rightarrow \alpha^2 = 7 \Rightarrow \alpha = \pm\sqrt{7}$

5. If A is a square matrix of order 3 and $|A| = 5$, then the value of $|2A'|$ is

(a) – 10 (b) 10 (c) – 40 (d) 40

[CBSE 2020]

Solutions

(d) : Given, A is a 3 × 3 matrix and |A| = 5

Now, $|2a'| = 2^3|A'| = 8 \times 5 = 40$

6. Given that A is a square matrix of order 3 and $|A| = -4$, then $|adj\ A|$ is equal to

(a) – 4 (b) 4 (c) – 16 (d) 16

[CBSE SQP 2021-22]

Solutions

(d) : We know that, $|adj\ A| = |A|^{n-1}$, where n is the order of A.

Here, $|adj\ A| = (-4)^2 = 16$

7. If A is a square matrix of order 3 with |A| = 9, then write the value of |2 . adj A|.

[CBSE 2019]

Solutions

Given, |A| = 9

$\therefore \quad |2\ adj\ A| = 2^3(9)^2 = 8 \times 81 = 648.$

8. Find the equation of the line joining A(1, 3) and B(0, 0) using determinants and find the value of k if D(k, 0) is a point such that area of ΔABD is 3 square units.

[CBSE 2013C]

Solutions

Using determinants, the line joining A(1, 3) and B(0, 0) is given by $\begin{vmatrix} x & y & 1 \\ 1 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix} = O$

$\Rightarrow \quad 1(3x - y) = 0 \Rightarrow y = 3x$

Now, D(k, 0) is a point s.t. area of ΔABD = 3 sq. units

$\therefore \quad \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ k & 0 & 1 \end{vmatrix} = 3$

$\Rightarrow \quad (0 + 3k) = \pm 6 \Rightarrow k = \pm 2$

9. Find the adjoint of the matrix A = $\begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ and hence show that A.(adj A) = $|A|I_3$.

[CBSE 2015]

Solutions

Here, A = $\begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

$\Rightarrow \quad |A| = \begin{vmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{vmatrix}$

$= -1(1 - 4) - (-2)(2 + 4) - 2(-4 - 2)$

$= 3 + 12 + 12 = 27$

Now, $A_{11} = -3, A_{12} = -6, A_{13} = -6,$

$A_{21} = 6, A_{22} = 3, A_{23} = -6,$

$A_{31} = 6, A_{32} = -6, A_{33} = 3$

$\therefore \quad adj\ A = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$

$\therefore \quad A.(adj\ A) = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}.\begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$

$= \begin{bmatrix} 27 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{bmatrix} = 27\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|I_3.$

10. If A = $\begin{bmatrix}1 & 2 & -3\\ 3 & 2 & -2\\ 2 & -1 & 1\end{bmatrix}$, then find A^{-1} and use it to solve the following system of the equations:

$x + 2y - 3z = 6, 3x + 2y - 2z = 3,$

$2x - y + z = 2$

[CBSE 2020]

Solutions

Given, A = $\begin{bmatrix}1 & 2 & -3\\ 3 & 2 & -2\\ 2 & -1 & 1\end{bmatrix}$

$|A| = \begin{vmatrix}1 & 2 & -3\\ 3 & 2 & -2\\ 2 & -1 & 1\end{vmatrix} = 1(2-2) - 2(3+4) - 3(-3-4)$

$= -14 + 21 = 7 \neq 0$

$\therefore$ A^{-1} exists

Now, $A_{11} = 0, A_{12} = -7, A_{13} = -7, A_{21} = 1, A_{22} = 7,$

$A_{23} = 5, A_{31} = 2, A_{32} = -7, A_{33} = -4$

$\therefore$ $adj\ A = \begin{bmatrix}0 & 1 & 2\\ -7 & 7 & -7\\ -7 & 5 & -4\end{bmatrix}$

$\Rightarrow$ $A^{-1} = \frac{1}{|A|}\ adj\ A = \frac{1}{7}\begin{bmatrix}0 & 1 & 2\\ -7 & 7 & -7\\ -7 & 5 & -4\end{bmatrix}$

The given system of equations is

$x + 2y - 3z = 6$
$3x + 2y - 2z = 3$
$2x - y + z = 2$

The system of equations can be written as AX = B

where A = $\begin{bmatrix}1 & 2 & -3\\ 3 & 2 & -2\\ 2 & -1 & 1\end{bmatrix}; X = \begin{bmatrix}x\\ y\\ z\end{bmatrix}, B = \begin{bmatrix}6\\ 3\\ 2\end{bmatrix}$

$\because$ A^{-1} exists, so system of equations has a unique solution given by $X = A^{-1}B$

$\Rightarrow$ $\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \frac{1}{7}\begin{bmatrix}0 & 1 & 2\\ -7 & 7 & -7\\ -7 & 5 & -4\end{bmatrix}\begin{bmatrix}6\\ 3\\ 2\end{bmatrix} = \frac{1}{7}\begin{bmatrix}7\\ -35\\ -35\end{bmatrix} = \begin{bmatrix}1\\ -5\\ -5\end{bmatrix}$

$\Rightarrow$ $x = 1, y = -5, z = -5$

11. Use product $\begin{bmatrix}1 & -1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{bmatrix}\begin{bmatrix}-2 & 0 & 1\\ 9 & 2 & -3\\ 6 & 1 & -2\end{bmatrix}$ to solve the system of equations $x + 3z = 9, -x + 2y - 2z = 4, 2x - 3y + 4z = -3$

[CBSE 2017]

Solutions

We have, $\begin{bmatrix}1 & -1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{bmatrix}\begin{bmatrix}-2 & 0 & 1\\ 9 & 2 & -3\\ 6 & 1 & -2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = I$

$\Rightarrow \quad \begin{bmatrix}1 & -1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{bmatrix}^{-1} = \begin{bmatrix}-2 & 0 & 1\\ 9 & 2 & -3\\ 6 & 1 & -2\end{bmatrix}$ *or* $\begin{bmatrix}1 & 0 & 3\\ -1 & 2 & -2\\ 2 & -3 & 4\end{bmatrix}^{-1} = \begin{bmatrix}-2 & 9 & 6\\ 0 & 2 & 1\\ 1 & -3 & -2\end{bmatrix}$

Now the given system of equations is

$x + 3z = 9$

$-x + 2y - 2z = 4$

$2x - 3y + 4z = -3$

The system of equations can be written as AX = B

where, A = $\begin{bmatrix}1 & 0 & 3\\ -1 & 2 & -2\\ 2 & -3 & 4\end{bmatrix}, X = \begin{bmatrix}x\\ y\\ z\end{bmatrix}, B = \begin{bmatrix}9\\ 4\\ -3\end{bmatrix}$

Since, A^{-1} exists, so system of equations has a unique solution, given by

$X = A^{-1}B$

$\Rightarrow \quad \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}-2 & 9 & 6\\ 0 & 2 & 1\\ 1 & -3 & -2\end{bmatrix}\begin{bmatrix}9\\ 4\\ 3\end{bmatrix} = \begin{bmatrix}-18 + 36 - 18\\ 8 - 3\\ 9 - 12 + 6\end{bmatrix} = \begin{bmatrix}0\\ 5\\ 3\end{bmatrix}$

$\Rightarrow \quad x = 0, y = 5, z = 3$

12. The monthly incomes of Aryan and Babban are in the ratio 3 : 4 and their monthly expenditures are in the ratio 5 : 7. If each saves Rs. 15,000 per month, find their monthly incomes using matrix method. This problem reflects which value?

[CBSE 2016]

Solutions

Let the monthly income of Aryan be Rs. 3x and that of Babban be Rs. 4x.

Also, let monthly expenditure of Aryan be Rs. 5y and that of Babban be Rs. 7y.

According to question,

$$3x - 5y = 15000$$

$$4x - 7y = 15000$$

These equations can be written as AX = B

Where, A = $\begin{bmatrix} 3 & -5 \\ 4 & -7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$

$|A| = \begin{vmatrix} 3 & -5 \\ 4 & -7 \end{vmatrix} = (-21 + 20) = -1 \neq 0$

Thus, A^{-1} exist. So, system of equations has a unique solution and given by $X = A^{-1}B$

Now, adj (A) = $\begin{bmatrix} -7 & 5 \\ -4 & 3 \end{bmatrix}$

$\therefore \quad A^{-1} = \frac{adj(A)}{|A|} = \begin{bmatrix} 7 & -5 \\ 4 & -3 \end{bmatrix}$

Now, $X = A^{-1}B = \begin{bmatrix} 7 & -5 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 30000 \\ 15000 \end{bmatrix} \Rightarrow x = 30000 \text{ and } y = 15000$

So, monthly income of Aryan = 3 × 30000

= Rs. 90000

Monthly income of Babban = 4 × 30000 = Rs. 120000

From this question we are encouraged to save a part of money every month.

13. A trust invested some money in two type of bonds. The first bond pays 10% interest and second bond pays 12% interest. The trust received Rs. 2,800 as interest. However, if trust had interchanged money in bonds they would have got Rs. 100 less as interest. Using matrix method, find the amount invested by the trust. Interest received on this amount will be given to Helpage India as donation. Which value is reflected in this question?

Solutions

Let Rs. x be invested in the first bond and Rs. y be invested in the second bond.

According to question,

$\frac{10x}{100} + \frac{12y}{100} = 2800 \Rightarrow 10x + 12y = 280000$ …(i)

If the rate of interest had been interchanged, then the toal interest earned is Rs. 100 less than the previous interest i.e., Rs. 2700.

$\therefore \quad \frac{12x}{100} + \frac{10y}{100} = 2700 \Rightarrow 12x + 10y = 270000$ ...(ii)

The system of equations (i) and (ii) can be represented as

AX = B where A = $\begin{bmatrix} 10 & 12 \\ 12 & 10 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 280000 \\ 270000 \end{bmatrix}$

$|A| = \begin{vmatrix} 10 & 12 \\ 12 & 10 \end{vmatrix} = 100 - 144 = -44 \neq 0$

Thus, A^{-1} exists. So, system of equations has a unique solution and given by X = $A^{-1}B$

$adj\ A = \begin{bmatrix} 10 & -12 \\ -12 & 10 \end{bmatrix}$

Now, $X = A^{-1}B \Rightarrow X = \frac{adjA}{|A|}B$

$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{(-44)} \begin{bmatrix} 10 & -12 \\ -12 & 10 \end{bmatrix} \begin{bmatrix} 280000 \\ 270000 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{(-44)} \begin{bmatrix} -440000 \\ -660000 \end{bmatrix} = \begin{bmatrix} 1000 \\ 15000 \end{bmatrix}$

$\Rightarrow \quad x = 10000$ and $y = 15000$

Therefore, Rs. 10,000 be invested in the first bond and Rs. 15,000 be invested in the second bond. Thus, the total amount invested by the trust = 10,000 + 15,000 = Rs. 25,000.

The interest received will be given to Helpage India as donation reflects the helping and caring nature of the trust.

14. A coaching institute of English (Subject) conducts classes in two batches I and II and fees for rich and poor children are different. In batch I, it has 20 poor and 5 rich children and total monthly collection is Rs. 9,000, whereas in batch II, it has 5 poor and 25 rich children and total monthly collection is Rs. 26,000. Using matrix method, find monthly fees paid by each child of two types. What values the coaching institute is inculcating in the society?

[CBSE 2016]

Solutions

Let the monthly fees paid by poor and rich children be Rs. x and Rs. y respectively.

For batch I : $20x + 5y = 9000$ …(i)

For batch II : $5x + 25y = 26000$ …(ii)

The system of equations (i) and (ii) can be written as

AX = B

where, A = $\begin{bmatrix}20 & 5\\ 5 & 25\end{bmatrix}, X = \begin{bmatrix}x\\ y\end{bmatrix}$ and B = $\begin{bmatrix}9000\\ 26000\end{bmatrix}$

$|A| = \begin{vmatrix}20 & 5\\ 5 & 25\end{vmatrix} = 500 - 25 = 475 \neq 0$

Thus, A^{-1} exists. So, the given system has a unique solution and it is given by $X = A^{-1}B$.

$adj\ A = \begin{bmatrix}25 & -5\\ -5 & 20\end{bmatrix}$

$\boldsymbol{A^{-1}} = \frac{\boldsymbol{adjA}}{|\boldsymbol{A}|} = \frac{1}{475}\begin{bmatrix}25 & -5\\ -5 & 20\end{bmatrix}$

Now, $X = A^{-1}B$

$\Rightarrow \quad \begin{bmatrix}x\\ y\end{bmatrix} = \frac{1}{475}\begin{bmatrix}25 & -5\\ -5 & 20\end{bmatrix}\begin{bmatrix}9000\\ 26000\end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix}x\\ y\end{bmatrix} = \frac{1}{475}\begin{bmatrix}95000\\ 475000\end{bmatrix} \Rightarrow \begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}200\\ 1000\end{bmatrix}$

$\Rightarrow \quad x = 200, y = 1000$

Hence, the monthly fees paid by each poor child is Rs. 200 and the monthly fees paid by each rich child is Rs. 1000.

By offering discount to the poor children, the coaching institute offers an unbiased chance for the development and enhancement of the weaker section of our society.

15. Two schools A and B decided to award prizes to their students for three values, team spirit, truthfulness and tolerance at the rate of Rs. x, Rs. y and Rs. z per student respectively. School A, decided to award a total of Rs. 1,100 for the three values to 3, 1 and 2 students respectively while school B decided to award Rs. 1,400 for the three values to 1, 2 and 3 students respectively. If one prize for all the three values together amount to Rs. 600 then

(i) Represent the above situation by a matrix equation after forming linear equations.

(ii) Is it possible to solve the system of equations so obtained using matrices?

(iii) Which value you prefer to be rewarded most and why?

[CBSE 2015C]

Solutions

(i) Given, value of prize for team spirit = Rs x

Value of prize for truthfulness = Rs. y

Value of prize for tolerance = Rs. z

Linear equation for School A is $3x + y + 2z = 1100$

Linear equation for School B is $x + 2y + 3z = 1400$

Linear equation for Prize is $x + y + z = 600$

The corresponding matrix equation is $PX = Q$

where, P = $\begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and Q = $\begin{bmatrix} 1100 \\ 1400 \\ 600 \end{bmatrix}$

(ii) Now $|P| = \begin{vmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix}$

$= 3.(2-3) - 1.(1-3) + 2.(1-2)$

$= -3 + 2 - 2 = -3 \neq 0$

Thus, P^{-1} exists. So, system of equations has unique solution and it is given by $X = P^{-1}Q$

Now, cofactors of elements of P are

$A_{11} = -1, A_{12} = 2, A_{13} = -1.$

$A_{21} = 1, A_{22} = 1, A_{23} = -2$

$A_{31} = -1, A_{32} = -7, A_{33} = 5$

$\therefore \quad adj\ P = \begin{bmatrix} -1 & 1 & -1 \\ 2 & 1 & -7 \\ -1 & -2 & 5 \end{bmatrix}$

$P^{-1} = \frac{1}{|P|}.adj\ P = -\frac{1}{3}\begin{bmatrix} -1 & 1 & -1 \\ 2 & 1 & -7 \\ -1 & -2 & 5 \end{bmatrix}$

Now, $X = P^{-1}Q$

$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -1 & 1 & -1 \\ 2 & 1 & -7 \\ -1 & -2 & 5 \end{bmatrix}\begin{bmatrix} 1100 \\ 1400 \\ 600 \end{bmatrix}$

$= -\frac{1}{3}\begin{bmatrix} -300 \\ -600 \\ -900 \end{bmatrix} = \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix} \Rightarrow x = 100, y = 200, z = 300.$

Thus, the above system of equations is solvable.

(iii) The value truthfulness should be rewarded the most because a student who is truthful will be also tolerant and will work with a team spirit in the school.

16. A shopkeeper has 3 varieties of pens 'A', 'B' and 'C. Meenu purchased 1 pen of each variety for a total of Rs. 21. Jeevan purchased 4 pens of 'A' variety, 3 pens of 'B' variety and 2 pens

of 'C' variety for Rs. 60. While Shikha purchased 6 pens of 'A' variety, 2 pens of "B" variety and 3 pens of 'C' variety for Rs. 70. Using matrix method, find cost of each variety of pen.

[CBSE 2016]

Solutions

Let one pen of variety 'A' costs Rs. x, one pen of variety 'B' costs Rs. y and one pen of variety 'C' costs Rs. z.

According to question,

$x + y + z = 21$ (For Meenu)

$4x + 3y + 2z = 60$ (For Jeevan)

$6x + 2y + 3z = 70$ (For Shikha)

The given system of equations can be written as AX = B

where, A = $\begin{bmatrix}1 & 1 & 1\\ 3 & 4 & 2\\ 6 & 2 & 3\end{bmatrix}$, $X = \begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and B = $\begin{bmatrix}21\\ 60\\ 70\end{bmatrix}$

$|A| = \begin{vmatrix}1 & 1 & 1\\ 4 & 3 & 2\\ 6 & 2 & 3\end{vmatrix} = 1(9-4) - 1(12-12) + 1(8-18) = -5 \neq 0$

∴ A^{-1} exists and system of equations has a unique solution given by $X = A^{-1}B$.

Now, $A_{11} = 5, A_{12} = 0, A_{13} = -10,$

$A_{21} = -1, A_{22} = -3, A_{23} = 4,$

$A_{31} = -1, A_{32} = 2, A_{33} = -1$

∴ $adj\ A = \begin{bmatrix}5 & -1 & -1\\ 0 & -3 & 2\\ -10 & 4 & -1\end{bmatrix}$

$\boldsymbol{A^{-1}} = \frac{\mathbf{1}}{|A|}\ (\boldsymbol{adj\ A}) = \frac{1}{(-5)}\begin{bmatrix}5 & -1 & -1\\ 0 & -3 & 2\\ -10 & 4 & -1\end{bmatrix}$

Now, $X = A^{-1}B$

$\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \frac{1}{(-5)}\begin{bmatrix}5 & -1 & -1\\ 0 & -3 & 2\\ -10 & 4 & -1\end{bmatrix}\begin{bmatrix}21\\ 60\\ 70\end{bmatrix} \Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \frac{1}{(-5)}\begin{bmatrix}-25\\ -40\\ -40\end{bmatrix}$

$\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}5\\ 8\\ 8\end{bmatrix} \Rightarrow x = 5, y = 8, z = 8$

∴ cost of 1 pen of variety 'A' = Rs. 5

Cost of 1 pen of variety 'B' = Rs. 8

Cost of 1 pen of variety 'C' = Rs. 8

17. Two schools P and Q want to award their selected students on the values of discipline, politeness and punctuality. The school P wants to award Rs. x each, Rs. y each and Rs. z each for the three respective values to its 3, 2 and 1 students with a total award money of Rs. 1,000, School Q wants to spend Rs. 1,500 to award its 4, 1 and 3 students on the respective values (by giving the same award money for the three values as before). If the total amount of awards for one prize on each value is Rs. 600, using matrices, find the award money for each value.

Apart from the above three values, suggest one more value for awards.

[CBSE 2014]

Solutions

According to question, we have

$3x + 2y + z = 1000$ …(i)

$4x + y + 3z = 1500$ …(ii)

$x + y + z = 600$ …(iii)

The given system of equations can be written as AX = B

where, $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1000 \\ 1500 \\ 600 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = -5 \neq 0$$

∴ A is invertible and system of equations has a unique solution given by X = $A^{-1}B$

Now, $A_{11} = -2, A_{12} = -1, A_{13} = 3,$

$A_{21} = -1, A_{22} = 2, A_{23} = -1,$

$A_{31} = 5, A_{32} = -5, A_{33} = -5$

$$\therefore \quad adj\ A = \begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{adj\ (A)}{|A|} = \frac{-1}{5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -1 \end{bmatrix}$$

Now, X = $A^{-1}B$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}\begin{bmatrix} 1000 \\ 1500 \\ 600 \end{bmatrix} = \frac{-1}{5}\begin{bmatrix} -500 \\ -1000 \\ -1500 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix} \Rightarrow x = 100, y = 200, z = 300$$

Hence the money awarded for discipline, politeness and punctuality are Rs. 100, Rs. 200 and Rs. 300 respectively.

Apart from the above three values schools can award children for sincerity.

18. Two schools A and B want to award their selected students on the values of sincerity, truthfulness and helpfulness. The school A wants to award Rs. x each. Rs. y each and Rs. z each for the three respective values to 3,2 and 1 students respectively with a total award money of Rs. 1,600. School B wants to spend Rs. 2,300 to award its, 4, 1 and 3 students on the respective values (by giving the same award money to the three values as before). If the total amount of award for one prize on each value is Rs. 900, using matrices, find the award money for each value. Apart from these three values, suggest one more value which should be considered for award.

[CBSE 2014]

Solutions

According to question, we have

$3x + 2y + z = 1600$ …(i)

$4x + y + 3z = 2300$ …(ii)

$x + y + z = 900$ …(iii)

The given system of equations can be written as AX = B

where A = $\begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1600 \\ 2300 \\ 900 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = -5 \neq 0$$

∴ A is invertible and system of equations has a unique solution given by X = $A^{-1}B$

Now, $A_{11} = -2, A_{12} = -1, A_{13} = 3,$

$A_{21} = -1, A_{22} = 2, A_{23} = -1,$

$A_{31} = 5,\ A_{32} = -5, A_{33} = -5$

$\therefore \quad adj\ A = \begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$

$\therefore \quad A^{-1} = \frac{adj\ (A)}{|A|} = \frac{-1}{5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$

Now, $X = A^{-1}B$

$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}\begin{bmatrix} 1600 \\ 2300 \\ 900 \end{bmatrix} = \frac{-1}{5}\begin{bmatrix} -1000 \\ -1500 \\ -2000 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 200 \\ 300 \\ 400 \end{bmatrix}$

$\Rightarrow \quad x = 200, y = 300, z = 400$

Hence, the money awarded for sincerity, truthfulness and helpfulness are Rs. 200, Rs. 300 and Rs. 400 respectively.

Apart from the above three values schools can award children for discipline.

19. Two schools P and Q want to award their selected students on the values of tolerance, kindness and leadership. The school P wants to award Rs. x each, Rs. y each and Rs. z each for the three respective values to its 3, 2 and 1 students respectively with a total award money of Rs. 2200. School Q wants to spend 3100 to award its 4, 1 and 3 students on the respective values (by giving the same award money for the three values as school P). If the total amount of award for one prize on each value is Rs. 1200, using matrices, find the award money for each value.

Apart from the above these three values, suggest one more value which should be considered for award.

[CBSE 2014]

Solutions

According to question, we have

$3x + 2y + z = 2200$ …(i)

$4x + y + 3z = 3100$ …(ii)

$x + y + z = 1200$ …(iii)

The given system of equations can be written as AX = B, where

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2200 \\ 3100 \\ 1200 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = -5 \neq 0$$

$\therefore$ A is invertible and system of equations has a unique solution given by $X = A^{-1}B$

Now, $A_{11} = -2, A_{12} = -1, A_{13} = 3,$

$A_{21} = -1, A_{22} = 2, A_{23} = -1,$

$A_{31} = 5, A_{32} = -5, A_{33} = -5$

$$\therefore \quad adj\ A = \begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

$$\therefore \ \boldsymbol{A^{-1}} = \frac{\boldsymbol{adj\ (A)}}{\boldsymbol{|A|}} = \frac{-1}{5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

Now, $X = A^{-1}B$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}\begin{bmatrix} 2200 \\ 3100 \\ 1200 \end{bmatrix} = \frac{-1}{5}\begin{bmatrix} -1500 \\ -2000 \\ -2500 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 300 \\ 400 \\ 500 \end{bmatrix}$$

$$\Rightarrow \quad x = 300, y = 400, z = 500$$

Hence, the money awarded for tolerance, kindness and leadership are Rs. 300, Rs. 400 and Rs. 500 respectively.

Apart from the above three values schools can award children for sincerity.

20. A total amount of Rs. 7,000 is deposited in three different savings bank accounts with annual interest rates of 5%, 8% and $8\frac{1}{2}$% respectively. The total annual interest from these three accounts is Rs. 550. Equal amounts have been deposited in the 5% and 8% savings accounts. Find the amount deposited in each of the three accounts, with the help of matrices.

[CBSE 2014C]

Solutions

Let Rs. x, Rs. y and Rs. z be deposited at the rates of interest 5%, 8% and $8\frac{1}{2}\%$ respectively.

According to question,

$x + y + z = 7000$

$x - y = 0$

$x.\frac{5}{100} + y.\frac{8}{100} + z.\frac{17}{2} \times \frac{1}{100} = 550$

$\Rightarrow \quad 10x + 16y + 17z = 110000$

The system of equations can be written as AX = B

where, A = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 10 & 16 & 17 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and B = $\begin{bmatrix} 7000 \\ 0 \\ 110000 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 10 & 16 & 17 \end{vmatrix} = 1(-17) - (17) + 1(16 + 10)$

$= -8 \neq 0$

$\therefore$ A^{-1} exists. So, system of equations has a unique solution and it is given by X = $A^{-1}B$

Now, $A_{11} = -17, A_{12} = -17, A_{13} = 26,$

$A_{21} = -1, A_{22} = 7, A_{23} = -6$

$A_{31} = 1, A_{32} = 1, A_{33} = -2$

$\therefore \quad adj\ A = \begin{bmatrix} -17 & -1 & 1 \\ -17 & 7 & 1 \\ 26 & -6 & -2 \end{bmatrix}$

and $A^{-1} = \frac{1}{|A|}\ adj\ A = \frac{1}{-8}\begin{bmatrix} -17 & -1 & 1 \\ -17 & 7 & 1 \\ 26 & -6 & -2 \end{bmatrix}$

Now, X = $A^{-1}B$

$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{8}\begin{bmatrix} -17 & -1 & 1 \\ -17 & 7 & 1 \\ 26 & -6 & -2 \end{bmatrix}\begin{bmatrix} 7000 \\ 0 \\ 110000 \end{bmatrix}$

$= -\frac{1}{8}\begin{bmatrix} -9000 \\ -9000 \\ -38000 \end{bmatrix} = \begin{bmatrix} 1125 \\ 1125 \\ 4750 \end{bmatrix}$

$\Rightarrow \quad x = 1125 = y, z = 4750$

21. Two schools, P and Q, want to award their selected students for the values of sincerity, truthfulness and hard work at the rate of Rs. x, Rs. y and Rs. z for each respective value per student. School P awards its 2, 3 and 4 students on the above respective values with a total prize money of Rs. 4,600, School Q wants to award its 3, 2 and 3 students on the respective values with a total award money of Rs. 4,100. If the total amount of award money for one prize on each value is Rs. 1,500, using matrices find the award money for each value. Suggest one other value which the school can consider for awarding the students.

[CBSE 2014C]

Solutions

According to question, we have

$x + y + z = 1500$

$2x + 3y + 4z = 4600$

$3x + 2y + 3z = 4100$

The system of equations can be written as AX = B

where, A = $\begin{bmatrix}1 & 1 & 1\\ 2 & 3 & 4\\ 3 & 2 & 3\end{bmatrix}$; $X = \begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and B = $\begin{bmatrix}1500\\ 4600\\ 4100\end{bmatrix}$

$|A| = \begin{vmatrix}1 & 1 & 1\\ 2 & 3 & 4\\ 3 & 2 & 3\end{vmatrix}$

$= 1(9-8) - 1(6-12) + 1(4-9)$

$= 1 + 6 - 5 = 2 \neq 0$

$\therefore$ A^{-1} exists and so, system of equations has a unique solution given by $X = A^{-1}B$

Now, $A_{11} = 1, A_{12} = 6, A_{13} = -5,$

$A_{21} = -1, A_{22} = 0, A_{23} = 1,$

$A_{31} = 1, A_{32} = -2, A_{33} = 1$

$\therefore \quad adj\ A = \begin{bmatrix}1 & -1 & 1\\ 6 & 0 & -2\\ -5 & 1 & 1\end{bmatrix}$

$A^{-1} = \frac{1}{|A|}.A = \frac{1}{2}\begin{bmatrix}1 & -1 & 1\\ 6 & 0 & -2\\ -5 & 1 & 1\end{bmatrix}$

Now, $X = A^{-1}B$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 6 & 0 & -2 \\ -5 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1500 \\ 4600 \\ 4100 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1000 \\ 800 \\ 1200 \end{bmatrix} = \begin{bmatrix} 500 \\ 400 \\ 600 \end{bmatrix} \Rightarrow x = 500; y = 400; z = 600.$$

Apart from sincerity, truthfulness and hard work, the schools can include an award for regularity.

22. A school wants to award its students for the value of honesty, regularity and hard work with a total cash award of Rs. 6,000. Three times the award money for hard work added to that given for honesty amounts to Rs. 11,000. The award money given for honesty and hard work together is double the one given for regularity. Represent the above situation algebraically and find the award money for each value, using matrix method. Apart from these values, namely, honesty, regularity and hard work, suggest one more value which the school must include for awards.

[CBSE 2013]

Solutions

Let the award money for honesty = Rs. X

Let the award money for regularity = Rs. Y

Let the award money for hard work = Rs. Z

According to question, we have

$x + y + z = 6000; 3z + x = 1100; x + z - 2y = 0$

The system of equations can be written as AX = B

where, A = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{vmatrix}$

$= 1(0 + 6) - 1(1 - 3) + 1(-2 - 0)$

$= 6 - (-2) - 2 = 6 \neq 0$

$\therefore$ A is invertible. So, the given system has a unique solution given by $X = A^{-1}B$

Now, $A_{11} = 6, A_{12} = 2, A_{13} = -2$

$A_{21} = -3, A_{22} = 0, A_{23} = 3$

$A_{31} = 3, A_{32} = -2, A_{33} = -1$

$\therefore \quad adj\ (A) = \begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$

$\boldsymbol{A^{-1}} = \frac{\mathbf{1}}{|\boldsymbol{A}|}\ \boldsymbol{adj}\ (\boldsymbol{A}) \frac{1}{6}\begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$

Now, $X = A^{-1}B$

$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}\begin{bmatrix} 600 \\ 11000 \\ 0 \end{bmatrix}$

$= \frac{1}{6}\begin{bmatrix} 3000 \\ 12000 \\ 21000 \end{bmatrix} = \begin{bmatrix} 500 \\ 2000 \\ 3500 \end{bmatrix} \Rightarrow x = 500, y = 2000, z = 3500$

One more value which the school can include for award is discipline.

23. The management committee of a residential colony decided to award some of its members (say x) for honesty, some (say y) for helping others and some others (say z) for supervising the workers to keep the colony neat and clean. The sum of all the awardees is 12. Three times the sum of awardees for cooperation and supervision added to two times the numbers of awardees for honesty is 33. If the sum of the number of awardees for honesty and supervision is twice the number of awardees for helping others, using matrix method, find the number of awardees of each category. Apart from these values, namely, honesty, cooperation and supervision, suggest one more value which the management of the colony must include for awards.

[CBSE 2013]

Solutions

According to question, we have $x + y + z = 12$

$2x + 3(y + z) = 33 \Rightarrow 2x + 3y + 3z = 33$

$x + z = 2y \Rightarrow x - 2y + z = 0$

The system of equations can be written as AX = B

Where, A = $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & -2 & 1 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $B = \begin{bmatrix} 12 \\ 33 \\ 0 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & -2 & 1 \end{vmatrix} = 1(3+6) - 1(2-3) + 1(-4-3)$$

$= 9 + 1 - 7 = 3 \neq 0$

$\therefore$ A^{-1} exists. So, system of equations has a unique solution and it is given by

$X = A^{-1}B$

Now, $A_{11} = 9, A_{12} = 1, A_{13} = -7,$

$A_{21} = -3, A_{22} = 0, A_{23} = 3, A_{31} = 0, A_{32} = -1, A_{33} = 1$

$$\therefore \quad adj\ A = \begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\ (adj\ A) = \frac{1}{3}\begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}$$

Now, $X = A^{-1}B$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}\begin{bmatrix} 12 \\ 33 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 9 \\ 12 \\ 15 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

$\Rightarrow \quad x = 3, y = 4, z = 5.$

The management of the colony can include the awards for those members of the colony who help for keeping the environment of the colony free from pollution.

24. Two institutions decided to award their employees for the three values of resourcefulness, competence and determination in the form of prizes at the rate of Rs. x, Rs. y and Rs. z respectively per person. The first institution decided to award respectively 4, 3 and 2 employees with a total prize money of Rs. 37,000 and the second institution decided to award respectively 5, 3 and 4 employees with a total prize money of Rs. 47,000. If all the three prizes per person together amount to Rs. 12,000 then using matrix method find the value of x, y and z.

What values are described in the question?

[CBSE 2013C]

Solutions

According to question, we have

$x + y + z = 12000$

$4x + 3y + 2z = 37000$

$5x + 3y + 4z = 47000$

The system of equations can be written as AX = B

Where, A = $\begin{bmatrix}1 & 1 & 1\\4 & 3 & 2\\5 & 3 & 4\end{bmatrix}$; $X = \begin{bmatrix}x\\y\\z\end{bmatrix}$; $B = \begin{bmatrix}12000\\37000\\47000\end{bmatrix}$

$|A| = \begin{vmatrix}1 & 1 & 1\\4 & 3 & 2\\5 & 3 & 4\end{vmatrix}$

$= 1(12 - 6) - 1(16 - 10) + 1(12 - 15)$

$= 6 - 6 - 3 = -3 \neq 0$

$\therefore$ A^{-1} exists. So, system of equations has a unique solution and it is given by

$X = A^{-1}B$

Now, $A_{11} = 6, A_{12} = -6, A_{13} = -3$

$A_{21} = -1, A_{22} = -1, A_{23} = 2,$

$A_{31} = -1, A_{32} = 2, A_{33} = -1$

$\therefore$ $adj\ A = \begin{bmatrix}6 & -1 & -1\\-6 & -1 & 2\\-3 & 2 & -1\end{bmatrix}$

$\boldsymbol{A^{-1} = \frac{1}{|A|}adj\ A} = -\frac{1}{3}\begin{bmatrix}6 & -1 & -1\\-6 & -1 & 2\\-3 & 2 & -1\end{bmatrix}$

Now, $X = A^{-1}B$

$\Rightarrow$ $\begin{bmatrix}x\\y\\z\end{bmatrix} = -\frac{1}{3}\begin{bmatrix}6 & -1 & -1\\-6 & -1 & 2\\-3 & 2 & -1\end{bmatrix}\begin{bmatrix}12000\\37000\\47000\end{bmatrix} = -\frac{1}{3}\begin{bmatrix}-12000\\-15000\\-9000\end{bmatrix} = \begin{bmatrix}4000\\5000\\3000\end{bmatrix}$

$\Rightarrow$ $x = 4000, y = 5000; z = 3000$

The values described in this question are resourcefulness, competence and determination.

25. For which value of x, are the determinants $\begin{vmatrix}2x & -3\\5 & x\end{vmatrix}$ and $\begin{vmatrix}10 & 1\\-3 & 2\end{vmatrix}$ equal?

(a) ± 3
(b) -3
(c) ± 2
(d) 2 R

[CBSE 2023]

Solution

(c): Since, $\begin{vmatrix}2x & -3\\5 & x\end{vmatrix} = \begin{vmatrix}10 & 1\\-3 & 2\end{vmatrix}$

$\Rightarrow 2x^2 + 15 = 20 + 3 \Rightarrow 2x^2 = 23 - 15 = 8 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

26. The value of the determinant $\begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix}$ is

(a) 47
(b) -79
(c) 49
(d) -51

[CBSE 2023]

Solution

(a): Let $|A| = \begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix}$

$|A| = 2(1-8) - 7(1-10) + 1(8-10) = 2(-7) - 7(-9) + 1(-2) = -14 + 63 - 2 = 47$

27. Show that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of θ.

[CBSE 2023]

Solution

We have, $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$

$= x(-x^2 - 1) - \sin\theta(-x\sin\theta - \cos\theta) + \cos\theta(-\sin\theta + x\cos\theta)$

$= -x^3 - x + x\sin^2\theta + \sin\theta\cos\theta - \sin\theta\cos\theta + x\cos^2\theta$

$= -x^3 - x + x(\sin^2\theta + \cos^2\theta) = -x^3 - x + x$

$= -x^3$, which is independent of θ.

28. The value of the cofactor of the element of second row and third column in the matrix $\begin{bmatrix} 4 & 3 & 2 \\ 2 & -1 & 0 \\ 1 & 2 & 3 \end{bmatrix}$ is

(a) 5
(c) -11
(D) -5
(d) 11

[CBSE 2023]

Solution

(b) : Let $A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & -1 & 0 \\ 1 & 2 & 3 \end{bmatrix}$

Minor M_{23} of an element a_{23} of a matrix A is given by

$M_{23} = \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} = 8 - 3 = 5$

$\because$ Required cofactor $C_{23} = -M_{23} = -5$

29. If A is a square matrix of order 3 and $|A| = 6$, then the value of $|\text{adj}A|$ is :

(a) 6
(b) 36
(c) 27
(d) 216

[CBSE 2023]

Solution

(b): Since, order of a square matrix A is 3 and $|A| = 6$

We know that $|\text{Adj}A| = |A|^{n-1} = 6^{3-1} = 6^2 = 36$

30. If $|A| = 2$, where A is a 2×2 matrix, then $|4A^{-1}|$ equals:

(a) 4

(b) 2

(c) 8

(d) $\frac{1}{32}$

[CBSE 2023]

Solution

(b): Given, $|A| = 2$, where A is a 2×2 matrix.

Now, $|4A^{-1}| = 4|A|^{-1}$

and we know that, $|A|^{-1} = \frac{1}{|A|}$

$$\therefore |4A^{-1}| = 4 \times \frac{1}{|A|} = 4 \times \frac{1}{2} = 2$$

31. If matrix $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$, find A^{-1} and hence solve the following system of linear equations :

$3x + 2y + z = 20004x + y + 3z = 2500x + y + z = 900$

[CBSE 2023]

Solution

Given, $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$

$\Rightarrow |A| = 3(1-3) - 2(4-3) + 1(4-1) = -6 - 2 + 3 = -5 \neq 0$

So, A^{-1} exists and $A^{-1} = \frac{1}{|A|}\text{adj}(A)$

Now, $\text{adj}A = \begin{bmatrix} -2 & -1 & 3 \\ -1 & 2 & -1 \\ 5 & -5 & -5 \end{bmatrix}^T = \begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$

$\Rightarrow A^{-1} = -\frac{1}{5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$

Given $3x + 2y + z = 2000$

$4x + y + 3z = 2500$

$x + y + z = 900$

32. Given system of equation can be written as $AX = B$,

where, $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 2000 \\ 2500 \\ 900 \end{bmatrix}$

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}\begin{bmatrix} 2000 \\ 2500 \\ 900 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{5}\begin{bmatrix} -4000 - 2500 + 4500 \\ -2000 + 5000 - 4500 \\ 6000 - 2500 - 4500 \end{bmatrix}$$

$$= \frac{-1}{5}\begin{bmatrix} -2000 \\ -1500 \\ -1000 \end{bmatrix} = \begin{bmatrix} 400 \\ 300 \\ 200 \end{bmatrix}$$

$\therefore\ x = 400, y = 300$ and $z = 200$

33. Let $f(x) = \begin{vmatrix} x^2 & \sin x \\ p & -1 \end{vmatrix}$, where p is a constant. The value of p for which $f'(0) = 1$ is

(a) R
(b) 1
(c) 0
(d) -1

[CBSE 2024]

Solution

(d): Given, $f(x) = \begin{vmatrix} x^2 & \sin x \\ p & -1 \end{vmatrix} = -x^2 - p\sin x$

$\Rightarrow f'(x) = -2x - p\cos x$

We have, $f'(0) = 1 \Rightarrow -2(0) - p\cos(0) = 1 \Rightarrow -p = 1 \Rightarrow p = -1$

34. If $A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 2 & 3 \\ 5 & 1 & -1 \end{bmatrix}$, then the value of $|A(\text{adj. } A)|$ is:

(a) 1001
(b) 101
(c) 10
(d) 1000

[CBSE 2024]

Solution

(d): Wehave, $A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 2 & 3 \\ 5 & 1 & -1 \end{bmatrix} \Rightarrow |A| = (-2)(-2-3) = 10$

As, $|A(\text{adj} \cdot A)| = |A||\text{adj}A| = |A||A|^{n-1} = |A|^n$

$\Rightarrow |A(\text{ adj} \cdot A)| = (10)^3 = 1000$

35. For the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ \lambda & 2 & 0 \\ 1 & -2 & 3 \end{bmatrix}$ to be invertible, the value of λ is

(a) 0
(b) 10
(c) R - [10]
(d) $R - |-10|$

[CBSE 2024]

Solution

(d): Given, $A = \begin{bmatrix} 2 & -1 & 1 \\ \lambda & 2 & 0 \\ 1 & -2 & 3 \end{bmatrix}$

For invertible matrix, $|A| \neq 0$

So, $2(6-0) + 1(3\lambda - 0) + 1(-2\lambda - 2) \neq 0$

$\Rightarrow 12 + 3\lambda - 2\lambda - 2 \neq 0 \Rightarrow \lambda + 10 \neq 0 \Rightarrow \lambda \neq -10$

$\therefore$ Required value of λ is $R - \{-10\}$.

36. If A is a square matrix of order 3 such that the value of $|\text{adj}A| = 8$, then the value of $|A'|$ is

(a) $\sqrt{2}$

(b) $-\sqrt{2}$

(c) 8

(d) $2\sqrt{2}$

[CBSE 2024]

Solution

(d): Given, $|\text{adj}A| = 8$

We know that $|\text{adj}A| = |A|^{n-1}, n$ is the order of matrix.

$\Rightarrow |A|^{3-1} = 8 \Rightarrow |A|^2 = 8 \Rightarrow |A| = 2\sqrt{2} \therefore |A^T| = 2\sqrt{2} [\because |A| = |A^T|]$

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Continuity and Differentiability

Previous Years Questions

1. If the function $f(x) = \begin{cases} 3x - 8 & if\ x \leq 5 \\ 2k & if\ x > 5 \end{cases}$ continuous, then the value of k is

(a) 2/7 (b) 7/2 (c) 3/7 (d) 4/7

[CBSE 2021-22]

Solution

(b): Since $f(x)$ is continuous at x = 5,

$\Rightarrow \lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = f(5)$

$\Rightarrow \quad 3(5) - 8 = 2k \ \Rightarrow 7 = 2k \ \Rightarrow k = \frac{7}{2}$

2. If $f(x) = \begin{cases} 1, & if \quad x \leq 3 \\ ax + b, & if \quad 3 < x < 5 \\ 7, & if \quad x \geq 5 \end{cases}$, find the values of a and b so that $f(x)$ is a continuous function.

[CBSE 2013C]

Solution

Continuity at x = 3

$\because$ $\boldsymbol{f(x)}$ **is continuous at** $\mathbf{x = 3}$

$\therefore \quad \boldsymbol{f(3) = \lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x)} \quad \ldots (\mathbf{1})$

Now, $f(3) = 1$

$\lim_{x \to 3^-} f(x) = \lim(h \to 0) f(3 - h) = \lim_{h \to 0} 1 = 1$

$\lim_{x \to 3^+} f(x) = \lim_{h \to 0} f(3 + h)$

$= \lim_{h \to 0} [a(3 + h) + b] = 3a + b$

$\therefore$ From $(1), 3a + b = 1 \quad \ldots(2)$

Continuity at x = 5

$\because \quad f(x)$ is continuous at x = 5

$\therefore \quad f(5) = \lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) \quad \ldots (3)$

Now, $f(5) = 7$

$\lim_{x \to 5^+} f(x) = \lim_{h \to 0} f(5 + h) = \lim_{h \to 0} 7 = 7$

$\lim_{x \to 5^+} f(x) = \lim_{h \to 0} f(5-h)$

$= \lim_{h \to 0}[a(5-h)+b] = 5a+b$

$\therefore$ From (3), $5a+b=7$ ...(4)

Solving (2) and (4) we get

$a = 3, b = -8.$

3. Find the value of k, for which

$$f(x) = \begin{cases} \frac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & if\ -1 \le x < 0 \\ \frac{2x+1}{x-1}, & if\ 0 \le x < 1 \end{cases}$$ is continuous at $x = 0$.

[CBSE 2013]

Solution

$\because f(x)$ is continuous at x= 0

$\therefore \lim_{x \to 0^+} f(x) = f(0) = \lim_{x \to 0^-} f(x)$ ... (1)

Now, $f(0) = \frac{2 \times 0+1}{0-1} = -1$

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} \frac{2h+1}{h-1} = -1$$

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} (0-h) = \lim_{h \to 0} \frac{\sqrt{1-kh}-\sqrt{1+kh}}{-h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1-kh}-\sqrt{1+kh}}{-h} \times \frac{\sqrt{1-kh}+\sqrt{1+kh}}{\sqrt{1-kh}+\sqrt{1+kh}}$$

$$= \lim_{h \to 0} \frac{(1-kh)-(1+kh)}{-h\left[\sqrt{1-kh}+\sqrt{1+kh}\right]}$$

$$= \lim_{h \to 0} \frac{2k}{\sqrt{1-kh}+\sqrt{1-kh}} = \frac{2k}{2} = k$$

$\therefore$ From (1), we get $k = -1$

4. The value of k(k < 0) for which the function f defined as

$$f(x) = \begin{cases} \frac{1-\cos kx}{x \sin x}, & x \neq 0 \\ \frac{1}{2}, & x = 0 \end{cases}$$ is continuous at x= 0 is

(a) ± 1 (b) – 1 (c) $\pm \frac{1}{2}$ (d) $\frac{1}{2}$

[CBSE SQP 2021-22]

Solution

(b) : We have, $f(x) = \begin{cases} \frac{1-\cos kx}{x \sin x}, & x \neq 0 \\ \frac{1}{2}, & x = 0 \end{cases}$

$\because$ $f(x)$ is continuous at x = 0

$\therefore \lim_{x \to 0} \frac{1-\cos kx}{x \sin x} = \frac{1}{2} \Rightarrow \lim_{x \to 0} \frac{2 \sin^2 \frac{kx}{2}}{x^2 \frac{\sin x}{x}} = \frac{1}{2}$

$\Rightarrow \lim_{x \to 0} 2.\frac{k^2}{4} \left\{ \frac{\sin\left(\frac{kx}{2}\right)}{\frac{kx}{2}} \right\}^2 \frac{1}{\frac{(\sin x)}{2}} = \frac{1}{2} \Rightarrow \frac{k^2}{2} = \frac{1}{2} \Rightarrow k = \pm 1$

But k < 0 $\therefore k = -1$

5. Find the values of p and q, for which

$f(x) = \begin{cases} \frac{1-\sin^3 x}{3\cos^2 x}, & if\ x < \pi/2 \\ p, & if\ x = \pi/2 \\ \frac{q(1-\sin x)}{(\pi-2x)^2}, & if\ x > \pi/2 \end{cases}$ is continuous at $x = \pi/2$.

Solution

$\because$ $f(x)$ is continuous at $\pi/2$.

$\therefore \lim_{x \to \pi/2^-} f(x) = \lim_{x \to \pi/2^+} f(x) = f(\pi 2)$ ... (1)

Now, $\lim_{x \to \pi/2^-} f(x) = \lim_{h \to 0} f\left(\frac{\pi}{2} - h\right)$

$= \lim_{h \to 0} \frac{1 - \sin^3\left(\frac{\pi}{2} - h\right)}{3\cos^2\left(\frac{\pi}{2} - h\right)} = \lim_{h \to 0} \frac{1 - \cos^3 h}{3 \sin^2 h}$

$= \lim_{h \to 0} \frac{(1 - \cos h)(1 + \cos^2 h + \cos h)}{3(1 - \cos h)(1 + \cos h)}$

$= \lim_{h \to 0} \frac{(1 + \cos^2 h + \cos h)}{3(1 + \cos h)} = \frac{1 + 1 + 1}{3(1 + 1)} = \frac{1}{2}$

$and \lim_{x \to \pi/2^+} f(x) = \lim_{h \to 0} \left(\frac{\pi}{2} + h\right)$

$= \lim_{h \to 0} \frac{q\left[1 - \sin\left(\frac{\pi}{2} + h\right)\right]}{\left[\pi - 2\left(\frac{\pi}{2} + h\right)\right]^2} = \lim_{h \to 0} \frac{q(1 - \cos h)}{4h^2}$

$= \frac{q}{4} \times \lim_{h \to 0} \frac{2\sin^2 \frac{h}{2}}{\frac{h^2}{4} \times 4} = \frac{q}{4} \times \frac{2}{4} = \frac{q}{8}$ and $f(\pi/2) = p \quad \therefore \frac{1}{2} = \frac{q}{8} = p$

$= \quad p = \frac{1}{2}$ and $q = 4$ $[From\ (1)]$

6. Find the value of the constant k so that the function f, defined below, is continuous at $x = 0$, where

$$f(x) = \begin{cases} \left(\frac{1-\cos 4x}{8x^2}\right), & if\ x \neq 0 \\ k\ , & if\ x = 0 \end{cases}$$

[CBSE 2014C]

Solution

$\because f(x)$ is continuous at x = 0.

$\therefore \quad f(0) = k$

and $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1 - \cos 4x}{8x^2}$

$= \lim_{x \to 0} \frac{1 - \sin^2 2x}{8x^2} = \lim_{x \to 0} \left(\frac{\sin 2x}{2x}\right)^2 = 1$

$\because$ f is continuous at x = 0

$\therefore \quad f(0) = \lim_{x \to 0} f(x) \Rightarrow k = 1$

7. If $f(x) = \begin{cases} \frac{1-\cos 4x}{x^2}, & when\ x < 0 \\ a, & when\ x = 0 \\ \frac{\sqrt{x}}{\left(\sqrt{16+\sqrt{x}}\right)-4} & when\ x > 0 \end{cases}$

and f is continuous at $x = 0$, find the value of a.

[CBSE 2013C]

Solution

$\because f(x)$ is continuous at x = 0.

$\therefore \lim_{x \to 0^+} f(x) = f(0) = \lim_{x \to 0^-} f(x)$ ... (1)

Now, $f(0) = a$

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} \frac{\sqrt{h}}{\sqrt{16 + \sqrt{h} - 4}}$$

$$= \lim_{h \to 0} \frac{\sqrt{h}}{\sqrt{16+\sqrt{h}-4}} \times \frac{\sqrt{16+\sqrt{h}+4}}{\sqrt{16+\sqrt{h}+4}}$$

$$= \lim_{h \to 0} \frac{\sqrt{h}\left(\sqrt{16+\sqrt{h}+4}\right)}{16+\sqrt{h}-4^2} = \lim_{h \to 0}\left(\sqrt{16+\sqrt{h}}+4\right) = 8$$

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} \frac{1-\cos 4\,(-h)}{(-h)^2}$$

$$= \lim_{h \to 0} \frac{1-\cos 4h}{h^2} = \lim_{h \to 0} \frac{2\sin^2 2h}{h^2}$$

$$= 8.\lim_{h \to 0}\left(\frac{\sin 2h}{2h}\right)^2 = 8$$

∴ From (1), we get $a = 8$

Based on Differentiability

8. The function $f(x) = \begin{cases} x^2 & for\ x < 1 \\ 2-x & for\ x \geq 1 \end{cases}$

(a) not differentiable at x = 1

(b) differentiable at x = 1

(c) not continuous at x = 1

(d) neither continuous nor differentiable at x = 1

[CBSE 2021-22]

Solution

(a) : At x = 1

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} x^2 = 1$$

$$And\ \lim_{x \to 1^+} f(x) = \lim_{x \to 1} 2-x = 1$$

$$Also, f(1) = 2-1 = 1$$

$$\because \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(x)$$

∴ $f(x)$ is continuous at x =1

$$Now, L.H.D. = \lim_{x \to 1^-} \frac{f(x)-f(1)}{x-1}$$

$$= \lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1}(x+1) = 2$$

$R.H.D. = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{(2 - x) - 1}{x - 1} = -1$

$\because \quad L.H.D. \neq R.H.D.$

$\therefore \quad f(x)$ is not differentiable at x= 1

9. Show that the function $f(x) = |x - 3|, x \in R$, is continuous but not differentiable at $x = 3$.

[CBSE 2013]

Solution

$$f(x) = |x - 3| = \begin{cases} x - 3, & if\ x \geq 3 \\ -(x - 3), & if\ x < 3 \end{cases}$$

We have $f(3) = |3 - 3| = 0$

$\lim_{x \to 3^+} f(x) = \lim_{h \to 0} f(3 + h) = \lim_{h \to 0} (3 + h) = \lim_{h \to 0} h = 0$

$\lim_{x \to 3^-} f(x) = \lim_{h \to 0} f(3 - h) = \lim_{h \to 0} [-(3 - h - 3)] = \lim_{h \to 0} h = 0$

$\because \lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} f(x) = f(3) = 0$

So, $f(x)$ is continuous at x = 3.

Now, $Rf'(3) = \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h}$

$= \lim_{h \to 0} \frac{(3 + h - 3) - 0}{h} = \lim_{h \to 0} \frac{h}{h} = 1$

$And\ Lf'(3) = \lim_{h \to 0} \frac{f(3 - h) - f(3)}{-h}$

$= \lim_{h \to 0} \frac{[-(3 - h - 3)] - 0}{-h} = \lim_{h \to 0} \frac{h}{-h} = -1$

Thus, $Rf'(3) \neq Lf'(3)$

$\therefore \quad f(x)$ is not differentiable at x = 3.

10. The greatest integer function defined by $f(x) = [x],\ 0 < x < 2$ is not differentiable at x = _____.

[CBSE 2020]

Solution

The greatest integer function defined by $f(x) = [x], 0 < x < 2$ is not differentiable at x = 1.

$$x = 1$$

11. Find $\frac{dy}{dx}$ at $x = 1, y = \frac{\pi}{4}$ if $\sin^2 y + \cos xy = K$.

[CBSE 2017]

Solution

We have, $\sin^2 y + \cos xy = K$

$$\frac{2\sin y \cos y\, dy}{dx} + (-\sin xy)\left(x\frac{dy}{dx} + y\right) = 0$$

$(\because \textbf{Product rule} : (\boldsymbol{\mu v})' = \boldsymbol{\mu' v + \mu v'})$

$$\Rightarrow \frac{dy}{dx} = \frac{y\sin xy}{\sin 2y - x\sin xy}$$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{\left(1,\frac{\pi}{4}\right)} = \frac{\frac{\pi}{4}\sin\frac{\pi}{4}}{\sin\frac{\pi}{2} - \sin\frac{\pi}{4}} = \frac{\pi}{4\left(\sqrt{2}-1\right)}$$

12. If $\cos y = x\cos(a+y)$, where $\cos a \neq \pm 1$, prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$.

[CBSE 2014]

Solution

We have $\cos y = x\cos(a+y)$

$$\Rightarrow \quad x = \frac{\cos y}{\cos(a+y)}$$

Differentiating w.r.t. y on both sides, we get

$$\frac{dx}{dy} = \frac{\cos(a+y)\left(\frac{d}{dy}\cos y\right) - \cos y\left(\frac{d}{dy}\cos(a+y)\right)}{\cos^2(a+y)}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\cos(a+y)(-\sin y) + \cos y\sin(a+y)}{\cos^2(a+y)}$$

$$\Rightarrow \frac{dx}{dy} = \frac{\cos y\sin(a+y) - \cos(a+y)\sin y}{\cos^2(a+y)}$$

$$= \frac{\sin[(a+y)-y]}{\cos^2(a+y)} = \frac{\sin a}{\cos^2(a+y)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

13. If $y = \sin^{-1}\left\{x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right\}$ and $0 < x < 1$, then find $\frac{dy}{dx}$.

[CBSE 2014C]

Solution

We have, $y = \sin^{-1}\left(x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right)$

$\Rightarrow y = \sin^{-1}\left(x\sqrt{1-\left(\sqrt{x}\right)^2} - \sqrt{x}\sqrt{1-x^2}\right)$

$\Rightarrow \quad y = \sin^{-1} x - \sin^{-1}\sqrt{x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{a-x^2}} - \frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}} \cdot \frac{d}{dx}\left(\sqrt{x}\right)$$

$$\left(\because \frac{\boldsymbol{d}}{\boldsymbol{dx}}\left(\boldsymbol{\sin^{-1} x}\right) = \frac{\boldsymbol{1}}{\boldsymbol{\sqrt{1-x^2}}}\right)$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{x-x^2}}$$

14. If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, then prove that $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$.

[CBSE SQP 2019-20]

Solution

We have, $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$

Let $x = \sin A$, $y = \sin B$

$\therefore \sqrt{1-\sin^2 A} + \sqrt{1-\sin^2 B} = a(\sin A - \sin B)$

$\Rightarrow \cos A + \cos B = a(\sin A - \sin B)$

$\Rightarrow 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$= 2a\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$= \cos\left(\frac{A-B}{2}\right) = a\sin\left(\frac{A-B}{2}\right)$

$= \cot\left(\frac{A-B}{2}\right) = a \Rightarrow \frac{A-B}{2} = \cot^{-1} a$

$\Rightarrow A - B = 2\cot^{-1} a$

$\Rightarrow \sin^{-1} x - \sin^{-1} y = 2\cot^{-1} a$

Differentiating w.r.t. x, we get

$$\Rightarrow \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

15. If $x^m y^n = (x+y)^{m+n}$, prove that $\frac{dy}{dx} = \frac{y}{x}$.

[CBSE 2014]

Solution

Given $x^m y^n = (x+y)^{m+n}$

Taking log on both the sides, we get

$\log x^m + \log y^n = (m+n)\log(x+y$

$\Rightarrow \quad m\log x + n\log y = (m+n)\log(x+y)$

Differentiating w.r.t. x, we get

$$m.\frac{1}{x} + n.\frac{1}{y}.\frac{dy}{dx} = (m+n).\frac{1}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{n}{y} - \frac{m+n}{x+y}\right) = \frac{m+n}{x+y} - \frac{m}{x}$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{nx+ny-my-ny}{y(x+y)}\right) = \frac{mx+nx-mx-my}{x(x+y)}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{nx-my}{y(x+y)}\right) = \frac{nx-my}{x(x+y)} \quad \therefore \frac{dy}{dx} = \frac{y}{x}$$

16. If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$.

[CBSE 2013]

Solution

We have, $x^y = e^{x-y}$

Taking log on both sides, we get

$$y\log x = x - y$$

$$\Rightarrow \quad y(\log x + 1) = x$$

$$\Rightarrow \quad y = \frac{x}{\log x + 1}$$

Differentiating w.r.t. x on both sides, we get

$$\frac{dy}{dx} = \frac{(\log x + 1).1 - x\left(\frac{1}{x}\right)}{(1+\log x)^2}$$

$$= \frac{\log x + 1 - 1}{(1+\log x)^2}$$

$$= \frac{\log x}{(1+\log x)^2} \text{ (Hence, proved.)}$$

17. If $x = a(2\theta - \sin 2\theta)$ and $y = a(1 - \cos 2\theta)$, find $\frac{dy}{dx}$ when $\theta = \frac{\pi}{3}$.

[CBSE 2018]

Solution

We have, $x = a(2\theta - \sin 2\theta)$ ...(i)

and $y = a(1 - \cos 2\theta)$ ...(ii)

Differentiating (i) w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(2 - 2\cos 2\theta) \quad \text{... (iii)}$$

Differentiating (ii) w.r.t. θ, we get

$$\frac{dy}{d\theta} = 2a \sin 2\theta \quad \text{... (iv)}$$

Now, $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2a\sin 2\theta}{a(2-2\cos 2\theta)} = \frac{\sin 2\theta}{1-\cos 2\theta}$

$$\therefore \quad \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{\sin 2\left(\frac{\pi}{3}\right)}{1-\cos\frac{2\pi}{3}} = \frac{\sin\left(\pi-\frac{\pi}{3}\right)}{1-\cos\left(\pi-\frac{\pi}{3}\right)}$$

$$= \frac{\sin\left(\frac{\pi}{3}\right)}{1+\cos\left(\frac{\pi}{3}\right)} = \frac{\frac{\sqrt{3}}{2}}{1+\frac{1}{2}} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

18. If $x = \cos t\,(3 - 2\cos^2 t)$ and $y = \sin t\,(3 - 2\sin^2 t)$, find the value of $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$.

[CBSE 2014]

Solution

Here, $x = \cos t\,(3 - 2\cos^2 t), y = \sin t\,(3 - 2\sin^2 t)$

$$\Rightarrow \frac{dx}{dt} = -\sin t\,(3 - 2\cos^2 t) + \cos t\,[2.2\cos t \sin t]$$

$$= -3\sin t + 6\cos^2 t \sin t$$

$$and\ \frac{dy}{dt} = cos\ t\ (3 - 2\ sin^2\ t\ ) + sin\ t\ (-2.2\ sin\ t\ cos\ t)$$

$$= 3\cos t - 6\sin^2 t\ \cos t$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\cos t - 6\sin^2 t\cos t}{-3\sin t + 6\cos^2 t \sin t}$$

$$= \frac{3\cos t \,.\cos 2t}{3\sin t \,.\cos 2t} = \cot t$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = \cot\frac{\pi}{4} = 1$$

19. If y = $(\sin^{-1} x)^2$, prove that, $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0$.

[CBSE 2019]

Solution

We have, $y = (\sin^{-1} x)^2$ ...(i)

$$\Rightarrow \frac{dy}{dx} = 2(\sin^{-1} x) \times \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{4(\sin^{-1} x)^2}{(1-x^2)}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{4y}{1-x^2} \qquad [From\ (i)]$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 (1-x^2) = 4y \qquad \ldots(ii)$$

Again, differentiating (ii) w.r.t. x on both sides, we get

$$2\frac{dy}{dx}\frac{d^2y}{dx^2}(1-x^2) + \left(\frac{dy}{dx}\right)^2(-2x) = 4\left(\frac{dy}{dx}\right)$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0$$

20. If x = sin t, y = sin pt, prove that

$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$.

[CBSE 2019, 2016]

Solution

We have, $x = \sin t$ and $y = \sin pt$

$$\frac{dx}{dt} = \cos t \text{ and } p\cos pt$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{p\cos pt}{\cos t}$$

Differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{-p^2 \sin pt \cos t + p \cos pt \sin t}{\cos^2 t} \times \frac{dt}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{p^2 \sin pt \cos t + p \cos pt \sin t}{\cos^3 t}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{p^2 \sin pt \cos t}{\cos^3 t} + \frac{p \cos pt \sin t}{\cos^3 t}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-p^2 y}{\cos^2 t} + \frac{x\frac{dy}{dx}}{\cos^2 t}$$

$$\Rightarrow \cos^2 t \frac{d^2y}{dx^2} = -p^2 y + x\frac{dy}{dx}$$

$$\Rightarrow (1 - \sin^2 t)\frac{d^2y}{dx^2} = -p^2 y + x\frac{dy}{dx}$$

$$\Rightarrow (1 - x^2)\frac{d^{2y}}{dx^2} - x\frac{dy}{dx} + p^2 y = 0$$

21. If $y = e^{m \sin^{-1} x}, -1 \leq x \leq 1$, then show that $(1 - x^2)\frac{d^2y}{dx^2} - \frac{xdy}{dx} - m^2 y = 0$.

[CBSE 2015]

Solution

We have $y = e^{m \sin^{-1} x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = e^{m \sin^{-1} x}\left(\frac{m}{\sqrt{1 - x^2}}\right) = \frac{my}{\sqrt{1 - x^2}} \quad \ldots (1)$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = m\left[\frac{\sqrt{1 - x^2}\frac{dy}{dx} - \frac{y}{2\sqrt{1 - x^2}}.(-2x)}{(1 - x^2)}\right]$$

$$\Rightarrow (1 - x^2)\frac{d^2y}{dx^2} = m\left[my + \frac{xy}{\sqrt{1 - x^2}}\right] \qquad [From\ (1)]$$

$$\Rightarrow (1 - x^2)\frac{d^2y}{dx^2} = m\left[my + x.\left(\frac{1}{m}.\frac{dy}{dx}\right)\right]$$

$$\Rightarrow (1 - x^2)\frac{d^2y}{dx^2} = m^2 y + x\frac{dy}{dx}$$

$$\Rightarrow (1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2 y = 0$$

22. If y = $\left(x+\sqrt{1+x^2}\right)^n$, then show that $(1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=n^2y$.

[CBSE 2015, 2013C]

Solution

We have, $y=\left(x+\sqrt{1+x^2}\right)^n$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=n\left(x+\sqrt{1+x^2}\right)^{n-1}\left(1+\frac{2x}{2\sqrt{1+x^2}}\right)$$

$$\Rightarrow \frac{dy}{dx}=n\left(x+\sqrt{1+x^2}\right)^{n-1}\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{n\left(x+\sqrt{1+x^2}\right)^n}{\sqrt{1+x^2}}=\frac{ny}{\sqrt{1+x^2}} \quad \ldots(1)$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2}=n\left[\frac{\sqrt{1+x^2}.\frac{dy}{dx}-\frac{2(xy)}{2\sqrt{1+x^2}}}{1+x^2}\right]$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}=n\left[\sqrt{1+x^2}\times\frac{ny}{\sqrt{1+x^2}}-\frac{xy}{\sqrt{1+x^2}}\right]$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}=n^2y-\frac{nxy}{\sqrt{1+x^2}}$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}=n^2y-x\frac{dy}{dx} \qquad [From\ (1)]$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=n^2y$$

23. If $x=a(\cos 2\theta+2\theta\sin 2\theta)$ and $y=a(\sin 2\theta-2\theta\ cos\ 2\theta)$, find $\frac{d^2y}{dx^2}$ at $\theta=\frac{\pi}{8}$.

[CBSE SQP 2019-20]

Solution

Given, $x=a(\cos 2\theta+2\theta\sin 2\theta)$

$\Rightarrow \frac{dx}{d\theta} = a(-2\sin 2\theta + 2\sin 2\theta + 4\theta\cos 2\theta)$

$\Rightarrow \frac{dx}{d\theta} = a(4\theta\cos 2\theta)$

Again, $(y = a(\sin 2\theta - 2\theta\cos 2\theta)$

$\Rightarrow \frac{dy}{d\theta} = a(2\cos 2\theta + 4\theta\sin 2\theta - 2\cos 2\theta)$

$\Rightarrow \frac{dy}{d\theta} = a(4\theta\sin 2\theta)$ ... (ii)

From (i) and (ii), we get

$\frac{dy}{dx} = \frac{a(4\theta\sin 2\theta)}{a(4\theta\cos 2\theta)} = \tan 2\theta$

Differentiating again w.r.t. x, we get

$\frac{d^2y}{dx^2} = 2\sec^2 2\theta \cdot \frac{d\theta}{dx}$

$\Rightarrow \frac{d^2y}{dx^2} = 2\sec^2 2\theta \cdot \frac{1}{a(4\theta\cos 2\theta)}$

$$\left[\frac{d^2y}{dx^2}\right]_{\theta=\frac{\pi}{8}} = 2\sec^2\frac{\pi}{4}.\frac{1}{a\left(4\frac{\pi}{8}\cos\frac{\pi}{4}\right)} = \frac{8\sqrt{2}}{\pi a}$$

24. If y = x log $\left(\frac{x}{a+bx}\right)$, then prove that $x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$.

[CBSE 2013C]

Solution

Here, $y = x\log\left(\frac{x}{a+bx}\right)$ ...(1)

$\Rightarrow y = x\,[\log x - \log(a+bx)] = x\log x - x\log(a+bx)$

$\Rightarrow \frac{dy}{dx} = x.\frac{1}{x} + 1.\log x - \left[1.\log(a+bx) + x.\frac{1}{a+bx}.b\right]$

$= 1 - \frac{bx}{a+bx} + \log x - \log(a+bx)$

$\Rightarrow \frac{dy}{dx} = \frac{a}{a+bx} + \log\left(\frac{x}{a+bx}\right)$ ... (2)

$$\Rightarrow \frac{dy}{dx} = \frac{a}{a+bx} + \frac{y}{x} \qquad [Using\ (1)]$$

Again, differentiating (2) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = a.(-1)(a+bx)^{-2}.b + \frac{a(a+bx)}{x(a+bx)^2}$$

$$= \frac{-ab}{(a+bx)^2} + \frac{a}{x(a+bx)} = \frac{-abx + a(a+bx)}{x(a+bx)^2} = \frac{a^2}{x(a+bx)^2}$$

Now, $\text{R. H. S.} = \left(x\frac{dy}{dx} - y\right)^2$

$$= \left\{x.\left[\frac{a}{a+bx} + \frac{y}{x}\right] - y\right\}^2 = \left(\frac{ax}{a+bx}\right)^2 \text{ and L. H. S.} = x^3\frac{d^2y}{dx^2}$$

$$= \frac{a^2x^2}{(a+bx)^2} = \left[\frac{ax}{a+bx}\right]^2 = R.H.S.$$

25. Function f is defined as $f(x) = \begin{cases} 2x+2, & \text{if} \quad x < 2 \\ k, & \text{if} \quad x = 2 \\ 3x, & \text{if} \quad x > 2 \end{cases}$

Find the value of k for which the function f is continuous at $x = 2$.

[CBSE 2023]

Solution

We have, $f(x) = \begin{cases} 2x+2, & \text{if} \quad x < 2 \\ k, & \text{if} \quad x = 2 \\ 3x, & \text{if} \quad x > 2 \end{cases}$

$\text{LHL} = \lim_{x\to 2^-} f(x) = \lim_{x\to 2}(2x+2) = 2(2)+2 = 6$

$\text{RHL} = \lim_{x\to 2^+} f(x) = \lim_{x\to 2} 3x = 3(2) = 6$

Since, $f(x)$ is continuous at $x = 2$.

$\therefore$ LHL =RHL = f(2)

$\Rightarrow$ k = 6

26. The function $f(x) = |x|$ is

(a) continuous and differentiable everywhere.

(b) continuous and differentiable nowhere.

(c) continuous everywhere, but differentiable everywhere except at $x = 0$.

(d) continuous everywhere, but differentiable nowhere.

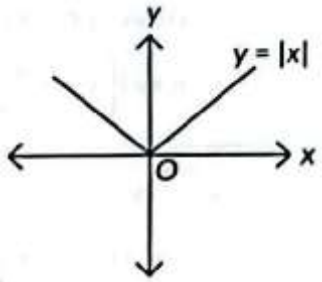
[CBSE 2023]

Solution

(c): $f(x) = |x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$

The function $f(x)$ is continuous everywhere but not differentiable at $x = 0$ as at $x = 0$

$$Lf'(0) = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{-x - 0}{x} = -1$$



$Rf'(0) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x-0}{x} = 1$

$\therefore\ Lf'(0) \neq Rf'(0)$, so $f(x)$ is not differentiable at $x = 0$.

27. If $f(x) = \begin{cases} x^2, & \text{if } x \geq 1 \\ x, & \text{if } x < 1 \end{cases}$ then show that f is not differentiable at $x = 1$.

[CBSE 2023]

Solution

We have, $f(x) = \begin{cases} x^2, & \text{if } x \geq 1 \\ x, & \text{if } x < 1 \end{cases}$

R.H.D. $= \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1}(x+1) = 2$

L.H.D. $= \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{x-1}{x-1} = 1$

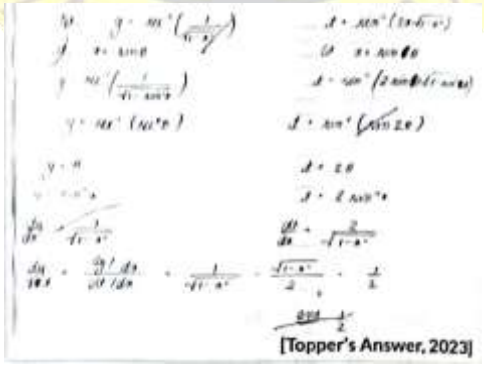
$\because$ L.H.D. $\neq$ R.H.D.

$\therefore f(x)$ is not differentiable at $x = 1$

28. Differentiate $\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$ w.r.t. $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$.

[CBSE 2023]

Solution


[Topper's Answer, 2023]

29. If $x = a\cos\theta + b\sin\theta, y = a\sin\theta - b\cos\theta$, then which one of the following is true?

(a) $y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$

(b) $y^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

(c) $y^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$

(d) $y^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}-y=0$

[CBSE 2023]

Solution

. (a): Given, $x=a\cos\theta+b\sin\theta$ and $y=a\sin\theta-b\cos\theta$

Differentiate w.r.t θ, we get

$\frac{dx}{d\theta}=-a\sin\theta+b\cos\theta$ and $\frac{dy}{d\theta}=a\cos\theta+b\sin\theta$

Now, $\frac{dy}{dx}=\frac{dy}{d\theta}\times\frac{d\theta}{dx}=\frac{a\cos\theta+b\sin\theta}{-(a\sin\theta-b\cos\theta)}=\frac{x}{-y}$

$$\Rightarrow\frac{dy}{dx}=\frac{-x}{y}\#(i)$$

Differentiate (i) w.r.t. x, we get

$$\frac{d^2y}{dx^2}=\frac{y(-1)+x(dy/dx)}{y^2}$$

$$\Rightarrow y^2\frac{d^2y}{dx^2}=-y+\frac{xdy}{dx}\Rightarrow y^2\frac{d^2y}{dx^2}-\frac{xdy}{dx}+y=0$$

30. If $y=\tan x+\sec x$, then prove that

$$\frac{d^2y}{dx^2}=\frac{\cos x}{(1-\sin x)^2}.$$

[CBSE 2023]

Solution

We have, $y=\tan x+\sec x$

Differentiating both sides w.r.t. x, we get $\frac{dy}{dx}=\sec^2x+\sec x\tan x$

$$\Rightarrow\frac{dy}{dx}=\frac{1}{\cos^2x}+\frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}=\frac{1+\sin x}{\cos^2x}$$

Now, again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2}=\frac{\cos^2x(\cos x)-(1+\sin x)(-2\cos x\sin x)}{(\cos^2x)^2}=\frac{\cos^3x+(1+\sin x)\cdot 2\sin x\cdot\cos x}{(1-\sin^2x)^2}$$

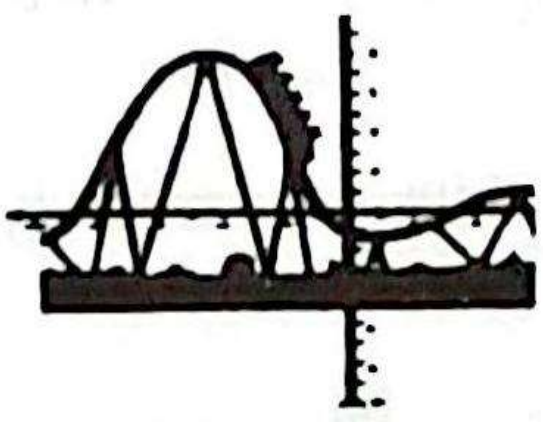
$$=\frac{\cos^3x+2\sin x\cos x+2\sin^2x\cos x}{(1-\sin x)^2(1+\sin x)^2}$$

$$=\frac{\cos x(\cos^2x+\sin^2x)+\sin^2x\cos x+2\sin x\cos x}{(1-\sin x)^2(1+\sin x)^2}$$

$$=\frac{\cos x[1+\sin^2x+2\sin x]}{(1-\sin x)^2(1+\sin x)^2}=\frac{\cos x\cdot(1+\sin x)^2}{(1-\sin x)^2(1+\sin x)^2}=\frac{\cos x}{(1-\sin x)^2}$$

Hence proved.

31. Case Study : The equation of the path traced by a roller-coaster is given by the polynomial $f(x)=a(x+9)(x+1)(x-3)$. If the roller-coaster crosses y-axis at a point $(0,-1)$, answer the following :



(i) Find the value of ' a '.

(ii) Find $f''(x)$ at $x = 1$.

[CBSE 2023]

Solution

(i) The given polynomial is

$$y = f(x) = a(x + 9)(x + 1)(x - 3) \#(i)$$

It is given that the roller-coaster crosses y-axis at $(0, -1)$ Put in (i), we get

$$-1 = a(0 + 9)(0 + 1)(0 - 3) \Rightarrow -27a = -1 \Rightarrow a = \frac{1}{27}$$

(ii) Now the polynomial is

$$f(x) = \frac{1}{27}(x + 9)(x + 1)(x - 3) \Rightarrow f(x) = \frac{1}{27}(x^3 + 7x^2 - 21x - 27)$$

On differentiating, we get

$$f'(x) = \frac{1}{27}(3x^2 + 14x - 21)$$

Again, on differentiating, we get

$$f''(x) = \frac{1}{27}(6x + 14)$$

At $x = 1$,

$$f''(1) = \frac{1}{27}(6 + 14) = \frac{20}{27}$$

32. For what value of k, the function given below is continuous at $x = 0$?

$$f(x) = \begin{cases} \dfrac{\sqrt{4 + x} - 2}{x} & , x \neq 0 \\ k & , x = 0 \end{cases}$$

(a) 0

(b) $\frac{1}{4}$

(c) 1

(d) 4

[CBSE 2024]

Solution

(b): As, $f(x) = \begin{cases} \frac{\sqrt{4+x}-2}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$ is continuous at $x = 0$

$\Rightarrow$ LHL = RHL = $f(0)$ or $\lim_{x\to 0} f(x) = f(0)$

$$\Rightarrow \lim_{x\to 0} \frac{\sqrt{4+x}-2}{x} \times \frac{\sqrt{4+x}+2}{\sqrt{4+x}+2} = k \Rightarrow \lim_{x\to 0} \frac{4+x-4}{x(\sqrt{4+x}+2)} = k \Rightarrow k$$

$$= \lim_{x\to 0} \frac{1}{(\sqrt{4+x}+2)} \therefore\ k = \frac{1}{4}$$

33. . If $y = \text{cosec}(\cot^{-1} x)$, then prove that $\sqrt{1+x^2}\frac{dy}{dx} - x = 0$.

[CBSE 2024]

Solution

$$y = \text{cosec}(\cot^{-1} x) \Rightarrow y = \text{cosec}\left(\text{cosec}^{-1}\sqrt{1+x^2}\right) \Rightarrow y = \sqrt{1+x^2} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{1+x^2}} \times 2x$$

$$\Rightarrow \sqrt{1+x^2}\frac{dy}{dx} - x = 0$$

Hence, proved.

34. . Check for differentiability of the function f defined by $f(x) = |x-5|$, at the point $x = 5$.

[CBSE 2024]

Solution

Given, $f(x) = |x-5|$

L.H.D. $= Lf'(5) = \lim_{x\to 5^-} \frac{f(x)-f(5)}{x-5} = \lim_{x\to 5^-} \frac{-x+5-0}{x-5} = -1$

R.H.D. $= Rf'(5) = \lim_{x\to 5^+} \frac{f(x)-f(5)}{x-5} = \lim_{x\to 5^+} \frac{x-5-0}{x-5} = 1$

$\therefore$ L.H.D. $\neq$ R.H.D. at $x = 5$

So, function $f(x)$ is not differentiable at $x = 5$.

35. If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, prove that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

[CBSE 2024]

Solution

We have, $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$

Put $x = \sin A, y = \sin B$ then, we get

$$\sqrt{1-\sin^2 A} + \sqrt{1-\sin^2 B} = a(\sin A - \sin B)$$

$$\Rightarrow \cos A + \cos B = a(\sin A - \sin B) \Rightarrow a = \frac{\cos A + \cos B}{\sin A - \sin B} = \frac{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}} \Rightarrow a = \cot\left(\frac{A-B}{2}\right)$$

$$\Rightarrow \cot^{-1} a = \frac{A-B}{2} \Rightarrow\ 2\cot^{-1} a = A - B$$

$$\Rightarrow\ 2\cot^{-1} a = \sin^{-1} x - \sin^{-1} y \#(i)$$

On differentiating (i) w.r.t. x, we get

$$\Rightarrow 0 = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} \Rightarrow \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$ (Hence proved)

36. If $x = e^{\cos 3t}$ and $y = e^{\sin 3t}$, prove that $\frac{dy}{dx} = -\frac{y\log x}{x\log y}$.

[CBSE 2024]

Solution

$x = e^{\cos 3t}$

Taking log on botk sides.

$\log_e x = \cos 3t$-(1)

Differentiating w.r.t.

$\frac{1}{x}\frac{dx}{dt} = -3\sin 3t$

So, $\frac{dx}{dt} = x(-3\sin 3t) - 0$

$y = e^{\sin 3t}$

Taking log on both sides.

$\log_e y \cdot \sin 3t -$ (3)

Differentiating /w.r.t.

$\frac{1}{y}dy = 3\cos 3t$

y dl

$\frac{dy}{dtdt} = y(3\cos 3t)$-(4)

(4) - (2)

$\frac{dy}{dx} = \frac{y(3\cos 3t)}{x(-3\sin 3t)} - \frac{y}{x}\frac{(\cos 3t)}{(\sin 3t)}$

As $\log x = \cos 3t$ and $\log y = \sin 3t$ from (1) and 0 ,

$\frac{dy}{dx} = \frac{-y}{x}\frac{(\log x)}{(\log y)} \sim \frac{-y\log x}{x\log y}$

Hence proved that

$$\frac{dy}{dx} = \frac{-y\log x}{x\log y}$$

[Topper's Answer, 2024]

Alternative Method

Given, $y = e^{\sin(3t)}$ and $x = e^{\cos(3t)}$

Taking logarithm on both sides, we get

$$\ln y = \ln e^{\sin(3t)}$$

$\Rightarrow \ln y = \sin(3t)$

Similarly, $\ln x = \cos(3t)$

Now, using $\sin^2(\theta) + \cos^2(\theta) = 1$

$\Rightarrow (\ln y)^2 + (\ln x)^2 = 1$

Differentiate w.r.t. x, we get

$$2\ln y \cdot \frac{1}{y}\frac{dy}{dx} + 2\ln x \cdot \frac{1}{x} = 0$$

$$\Rightarrow \frac{2}{y}\ln y \frac{dy}{dx} = \frac{-2}{x}\ln x \Rightarrow \frac{dy}{dx} = \frac{-2\ln x}{x}\frac{y}{2\ln y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y\ln x}{x\ln y}$$

Hence, proved.

37. If $y = (\tan^{-1} x)^2$, show that

$$(x^2+1)^2\frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2.$$

[CBSE 2024]

Solution

We have, $y = (\tan^{-1} x)^2$

On differentiating both sides of (i) w.r.t. x, we get

$$\frac{dy}{dx} = 2\tan^{-1} x \cdot \frac{d}{dx}(\tan^{-1} x) = 2\tan^{-1} x \cdot \frac{1}{1+x^2}$$

$$\Rightarrow (1+x^2)\frac{dy}{dx} = 2\tan^{-1} x$$

On squaring both sides, we get

$$(1+x^2)^2 \left(\frac{dy}{dx}\right)^2 = 4(\tan^{-1} x)^2 \#(ii)$$

$\Rightarrow (1+x^2)^2 \left(\frac{dy}{dx}\right)^2 - 4y = 0$ (Using (i))

On differentiating (ii) w.r.t. x, we get

$$(1+x^2)^2 \left(2\frac{dy}{dx}\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^2 (2(1+x^2)(2x)) - 4\frac{dy}{dx} = 0$$

$$\Rightarrow (1+x^2)^2 \frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} - 2 = 0$$

($\because$ Dividing by $2\frac{dy}{dx}$ both sides)

$$\Rightarrow (1+x^2)^2 \frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} = 2$$

Hence proved.

38. If $f(x) = \begin{cases} \frac{\log(1+ax)+\log(1-bx)}{x}, & \text{for } x \neq 0 \\ k, & \text{for } x = 0 \end{cases}$ is continuous at $x = 0$, then the value of k is :

(a) a
(b) $a + b$
(c) $a - b$
(d) b

[CBSE 2025]

Solution

(c) : As the function $f(x)$ is continuous at $x = 0$,

$$\lim_{x\to 0} \frac{\log(1+ax)+\log(1-bx)}{x} = k$$

$$\Rightarrow k = \lim_{x\to 0} a\frac{\log(1+ax)}{ax} + (-b)\frac{\log(1-bx)}{(-bx)} \Rightarrow k = a - b$$

$$\left[\because \lim_{x\to 0} \log\frac{(1+x)}{x} = 1\right]$$

39. Assertion (A) : $f(x) = \begin{cases} 3x-8, & x \leq 5 \\ 2k, & x > 5 \end{cases}$ is continuous at $x = 5$ for $k = \frac{5}{2}$.

Reason (R) : For a function **f** to be continuous at $x = a, \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$.

(a) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(b) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(c) Assertion (A) is true, but Reason (R) is false.
(d) Assertion (A) is false, but Reason (R) is true.

[CBSE 2025]

Solution

(d): Given, f is continuous at $x = 5$.

$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) \Rightarrow \text{Lt}_{x\to 5^-}(3x - 8) = 2k \Rightarrow 7 = 2k \Rightarrow k = \frac{7}{2}$

$\therefore$ Assertion is false and reason is true.

40. If $f(x) = |x| + |x - 1|$, then which of the following is correct?
(a) $f(x)$ is both continuous and differentiable, at $x = 0$ and $x = 1$.
(b) $f(x)$ is differentiable but not continuous, at $x = 0$ and $x = 1$.
(c) $f(x)$ is continuous but not differentiable, at $x = 0$ and $x = 1$
(d) $f(x)$ is neither continuous nor differentiable, at $x = 0$ and $x = 1$.

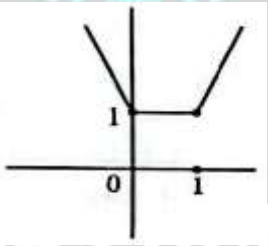
[CBSE 2025]

Solution

(c): We have, $f(x) = |x| + |x - 1|$

Now, $|x| = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$

and $|x - 1| = \begin{cases} x - 1, \text{ if } x \geq 1 \\ -x + 1, \text{ if } x < 1 \end{cases}$



$\therefore f(x) = |x| + |x - 1| = \begin{cases} -2x + 1, \text{if } x < 0 \\ 1, \text{if } 0 \leq x < 1 \\ 2x - 1, \text{if } x \geq 1 \end{cases}$

Clearly, from the graph, the function is continuous but not differentiable at $x = 0$ and $x = 1$.

41. Find $\frac{dy}{dx}$, if $y = x^{\tan x} + \frac{\sqrt{x^2+1}}{2}$.

[CBSE 2025]

Solution

We have, $y = x^{\tan x} + \frac{\sqrt{x^2+1}}{2}$

Let $v = x^{\tan x}$

Taking log on both sides, we get $\log v = \tan x \cdot \log x$

Differentiating w.r.t x, we get

$\frac{1}{v} \cdot \frac{dv}{dx} = \log x \cdot \sec^2 x + \frac{\tan x}{x}, \frac{dv}{dx} = v\left(\log x \sec^2 x + \frac{\tan x}{x}\right)$

$= x^{\tan x}\left(\log x \sec^2 x + \frac{\tan x}{x}\right) \#(ii)$

Now, differentiating (i) w.r.t. x, we get

$\frac{dy}{dx} = \frac{dv}{dx} + \frac{1}{2}\left(\frac{1}{2}\right)\frac{(2x)}{\sqrt{x^2+1}}\frac{dy}{dx} = x^{\tan x}\left(\log x \sec^2 x + \frac{\tan x}{x}\right) + \frac{x}{2\sqrt{x^2+1}}$

[From (ii)]

42. If $y = \log\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$, then show that $x(x+1)^2 y_2 + (x+1)^2 y_1 = 2$.

[CBSE 2025]

Solution

We have, $y = \log\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$

i.e. $y = 2\log\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$

On differentiation w.r.t. x, we get

$$y_1 = \frac{2}{\sqrt{x} + \frac{1}{\sqrt{x}}} \frac{d}{dx}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) = \frac{2\sqrt{x}}{x+1}\left(\frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}\right)$$

$$= \frac{1}{x+1}\left(1 - \frac{1}{x}\right) \Rightarrow y_1 = \frac{x-1}{x(x+1)} \quad \text{(i)}$$

Again, differentiating w.r.t. x, we get

$$y_2 = \frac{x(x+1) - (x-1)(2x+1)}{x^2(x+1)^2}$$

$$\Rightarrow y_2 = \frac{-x^2 + 2x + 1}{x^2(x+1)^2} \Rightarrow x(x+1)^2 y_2 = \frac{-x^2 + 2x + 1}{x} \quad \text{(ii)}$$

Now, $x(x+1)^2 y_2 + (x+1)^2 y_1$

$$= \frac{-x^2 + 2x + 1}{x} + \frac{(x+1)^2(x-1)}{x(x+1)}$$

[From (i) and (ii)]

$$= \frac{-x^2 + 2x + x^2}{x} = 2$$

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Application of Derivatives

1. A particle moves along the curve $x^2 = 2y$. At what point, ordinate increases at the same rate as abscissa increases?

[CBSE SQP 2019-20]

Solution

We have, $x^2 = 2y$ … (i)

$\Rightarrow \quad 2x\frac{dx}{dt} = 2\frac{dy}{dt} \qquad \left(Given, \frac{dy}{dt} = \frac{dx}{dt}\right)$

$\Rightarrow \quad 2x\frac{dx}{dt} = 2\frac{dx}{dt} \Rightarrow x = 1$

From (i), $y = \frac{1}{2}$

So, point is $\left(1, \frac{1}{2}\right)$.

2. The radius of a circle is increasing at the uniform rate of 3 cm/sec. At the instant when the radius of the circle is 2 cm, its area increases at the rate of_____ cm^2/s.

[CBSE 2020]

Solution

Let r be the radius and A be the area of circle. Given that $\frac{dr}{dt} = 3$ cm/sec …(i)

We know that, area of circle $A = \pi r^2$

$\therefore \quad \frac{dA}{dt} = 2\pi r\frac{dr}{dt} = 2\pi r.3$ [Using (i)]

$= 6\pi r$

$\therefore \quad \left(\frac{dA}{dt}\right)_{r=2\ cm} = 12\pi\ cm^2/s$

OR

$\frac{dr}{dt} = 3$

$A = Ar^2$

$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$

$\left(\frac{dA}{dt}\right)_{r=\partial} = 2\pi(2)(3) = (\partial\pi\ cm^2(\text{sec }).$

3. The total cost C(x) associated with provision of free mid-day meals to x students of a school in primary classes is given by

$C(x) = 0.005x^3 - 0.02x^2 + 30x + 50$

If the marginal cost is given by rate of change $\frac{dC}{dx}$ of total cost, write the marginal cost of food for 300 students. What value is shown here?

[CBSE 2013C]

Solution

Given, $C(x) = 0.005x^3 - 0.02x^2 + 30x + 50$

Marginal cost = $\frac{dC}{dx} = 0.015x^2 - 0.04x + 30$

Now, $\left(\frac{dC}{dx}\right)_{x=300} = 0.015 \times 300^2 - 0.04 \times 300 + 30$

= 1350- 12 + 30 = 1368.

The value indicated here is that a kind of care and concern is shown towards the health of students of primary classes by providing free mid-day meal to them.

4. The volume of a sphere is increasing at the rate of 3 cubic centimeter per second. Find the rate of Increase of its surface area, when the radius is 2 cm.

[CBSE 2017]

Solution

Let r, S and V respectively be the radius, surface area and volume of sphere at any time t.

$Then, \frac{dV}{dt} = 3\ cm^3/sec$

$Now, V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{3}{4\pi r^2}$

$Also, S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

$\Rightarrow \frac{dS}{dt} = \frac{6}{r}$

$\therefore \left(\frac{dS}{dt}\right)_{r=2} = \frac{6}{2} = 3\ cm^2/sec$

5. A ladder 13 m long is leaning against a vertical wall. The bottom of the ladder is dragged away from the wall along the ground at the rate of 2 cm/sec. How fast is the height on the wall decreasing when the foot of the ladder is 5 m away from the wall?

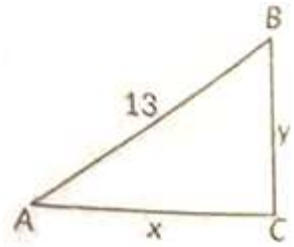
[CBSE 2019]

Solution

Let foot of the ladder is at a distance x from the wall and height on the wall is y.

Here, $x^2 + y^2 = (13)^2$ [Using Pythagoras theorem]

Differentiating with respect to t, we get



$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$\Rightarrow \frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}$

When x = 5 m, $y^2 = (13)^2 = 169 - 25 = 144$

$\therefore \quad y = 12\ m$

Also, $\frac{dx}{dt} = 2\ cm/sec$ $[Given]$

$\therefore \frac{dy}{dt} = \frac{-5}{12} \times 2 = \frac{-5}{6}\ cm/sec$

6. The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. Find the rate at which the area increases, when the side is 10 cm.

[CBSE 2014C]

Solution

Let 'a' be the side of an equilateral triangle.

Then $\frac{da}{dt} = 2\ cm/sec$

Let 'A' be the area of an equilateral triangle, then

$A = \frac{\sqrt{3}}{4}a^2 \Rightarrow \frac{dA}{dt} = 2 \times \frac{\sqrt{3}}{4}a\frac{da}{dt} = \frac{\sqrt{3}}{2}a\frac{da}{dt}$

$\therefore \quad \left(\frac{dA}{dt}\right)_{a=10} = \frac{\sqrt{3}}{2} \times 10 \times 2 = 10\sqrt{3}\ cm^2/sec.$

7. The real function f(x) = $2x^3 - 3x^2 - 36x + 7$ is

(a) Strictly increasing in $(-\infty, -2)$ and strictly decreasing in $(-2, \infty)$

(b) Strictly decreasing in (-2, 3)

(c) Strictly decreasing in $(-\infty, 3)$ and strictly increasing in $(3, \infty)$

(d) Strictly decreasing in $(-\infty, -2) \cup (3, \infty)$

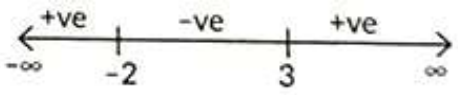
[CBSE SQP 2021-22]

Solution

(b): We have, f(x) = $2x^3 - 3x^2 - 36x + 7$

$\Rightarrow f'(x) = 6x^2 - 6x - 36 - 6(x^2 - x - 6)$

$= 6(x-3)(x+2)$



$\therefore f(x)$ is strictly increasing in $(-\infty, -2) \cup (3, \infty)$ and strictly decreasing in $(-2, 3)$.

8. Find the intervals in which $f(x) = sin\ 3x - cos\ 3x, 0 < x < \pi$, is strictly increasing or strictly decreasing.

[CBSE 2016]

Solution

$f(x) = \sin 3x - \cos 3x \Rightarrow f'(x) = 3\cos 3x + 3\sin 3x$

$f'(x) = 0 \Rightarrow 3\cos 3x = -3\sin 3x$

$\Rightarrow \cos 3x = -\sin 3x \Rightarrow \tan 3x = -1$

which gives $3x = \frac{3\pi}{4} or \frac{7\pi}{4} or \frac{11\pi}{4}$

$\Rightarrow \quad x = \frac{\pi}{4} or \frac{7\pi}{12} or \frac{11\pi}{12}$ $[\because 0 < x < \pi]$

Then points $x = \frac{\pi}{4}, x = \frac{7\pi}{12}$ and $x = \frac{11\pi}{12}$ divide the interval $(0, \pi)$into four disjoint intervals,

$\left(0, \frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{7\pi}{12}\right), \left(\frac{7\pi}{12}, \frac{11\pi}{12}\right), \left(\frac{11\pi}{12}, \pi\right)$

Now, $f'(x) > 0$ in $\left(0, \frac{\pi}{4}\right)$

$f'(x) < 0$ in $\left(\frac{\pi}{4}, \frac{7\pi}{12}\right)$

$\Rightarrow$ f is strictly decreasing in $\left(\frac{\pi}{4}, \frac{7\pi}{12}\right)$

$f'(x) > 0$ in $\left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$

$\Rightarrow$ f is striclty increasing in $\left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$

$f'(x) < 0$ in $\left(\frac{11\pi}{12}, \pi\right)$

$\Rightarrow$ f is strictly decreasing in $\left(\frac{11\pi}{12}, \pi\right)$

Hence, f is strictly increasing in the intervals

$\left(0, \frac{\pi}{4}\right) \cup \left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$

and f is strictly decreasing in the intervals

$$\left(\frac{\pi}{4}, \frac{7\pi}{12}\right) \cup \left(\frac{11\pi}{12}, \pi\right)$$

9. The area of a trapezium is defined by function f and given by $f(x) = (10 + x)\sqrt{100 - x^2}$, then the area when it is maximised is

(a) 75 cm^2 (b) $7\sqrt{3}$ cm^2 (c) $75\sqrt{3}$ cm^2 (d) 5 cm^2

[CBSE SQP 2021-22]

Solution

(c) : We have, $f(x) = (10 + x)\sqrt{100 - x^2}$

Which will give some real area if – 10 < x < 10

$$\Rightarrow f'(x) = \frac{(10 + x) \times (-2x)}{2\sqrt{100 - x^2}} + \sqrt{100 - x^2} \times 1$$

$$\Rightarrow \quad f'(x) = \frac{-2x^2 - 10x + 100}{\sqrt{100 - x^2}}$$

For critical points, put

$f'(x) = 0 \quad \Rightarrow x^2 + 5x - 50 = 0$

$\Rightarrow (x + 10)(x - 5) = 0$

$\Rightarrow x = -10 \; or \; 5 \; \Rightarrow x = 5 \qquad [\because -10 < x < 10]$

Now, $f''(x)$

$$= \frac{\left(\sqrt{100 - x^2}\right)(-4x - 10) + (2x^2 + 10x - 100) \times \frac{1}{2}\frac{(-2x)}{\sqrt{100 - x^2}}}{(100 - x^2)}$$

$$= \frac{2x^3 - 300x - 1000}{(100 - x^2)^{3/2}} \Rightarrow f''(5) = \frac{-30}{\sqrt{75}} < 0$$

$\therefore$ Maximum area of trapezium

$= (10 + 5)\left(\sqrt{75}\right) = 75\sqrt{3}\, cm^2$

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10. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of material will be least when depth of the tank is half of its width. If the cost is to be borne by nearby settled lower income families, for whom water will be provided, what kind of value is hidden in this question?

[CBSE 2018]

Solution

Let x be the side of square base and y be the height of the open tank.

$\therefore I = x, b = x$ and h = y

Where I, b and h be the length, breadth and height of tank respectively.

Volume of tank V = $x^2y \Rightarrow y = \frac{V}{x^2}$

The cost of the material will be least if the total surface area is least.

Total surface area of tank (S) = $x^2 + 4xy$

$\Rightarrow S = x^2 + 4x\left(\frac{V}{x^2}\right) \quad \left(\because y = \frac{V}{x^2}\right)$

$\Rightarrow \quad S = x^2 + \frac{4V}{x} \Rightarrow \frac{dS}{dx} = 2x - \frac{4V}{x^2}$

For maxima or minima, $\frac{dS}{dx} = 0$

$\Rightarrow 2x - \frac{4V}{x^2} = 0 \Rightarrow x^3 = 2V \Rightarrow x = 2y \quad (\because V = x^2y)$

Also, $\frac{d^2S}{dx^2} = 2 + \frac{8V}{x^3} > 0$

$\therefore$ Cost of material is least, when $y = \frac{x}{2}$

i.e., the depth of the tank is half of its width.

As the cost is borne by nearby settled lower income families it shows that they are spending money on social welfare so that no body will face the water problem in future. It shows social responsibility.

11. Show that the height of the right circular cylinder of greatest volume which can be inscribed in a right circular cone of height h and radius r is one-third of the height of the cone, and the greatest volume of the cylinder is $\frac{4}{9}$ times the volume of the cone.

[CBSE 2020]

Solution

We have, $\tan\theta = \frac{r}{h} = \frac{R}{h-H}$

$\Rightarrow R = r\left(1 - \frac{H}{h}\right)$

Now, volume of cylinder (V) = $\pi r^2 H = \pi r^2 \left(1 - \frac{H}{h}\right)^2 H$

For maximum volume, $\frac{dV}{dH} = 0$

$\Rightarrow \pi r^2 \left[H \times 2\left(1 - \frac{H}{h}\right) \times -\frac{1}{h} + \left(1 - \frac{H}{h}\right)^2\right] = 0$

$\Rightarrow \pi r^2 \left[-\frac{4H}{h} + \frac{3H^2}{h^2} + 1\right] = 0$

$\Rightarrow \pi r^2 \left(\frac{H}{h} - 1\right)\left(\frac{3H}{h} - 1\right) = 0$

$\Rightarrow H = \frac{h}{3}$ $[\because H \neq h]$

So, height of cylinder = $\frac{1}{3}$ of height of cone

Also, maximum volume of cylinder

$= \pi r^2 \left(1 - \frac{1}{3}\right)^2 \times \frac{h}{3} = \left(\frac{1}{3}\pi r^2 h\right) \times \frac{4}{9}$

$= \frac{4}{9}$ of volume of cone.

OR

Let the cone be represented by ABC and the cylinder have radius and height x and y respectively.

$\therefore$ As ΔAEF and ΔAGC are

Similar

$= \frac{h-y}{h} = \frac{x}{r}$

$rh - ry = xh \Rightarrow \frac{h - xh}{r} = y$

Volume of the cylinder $V = \pi x^2 y$

$\Rightarrow V = \frac{\pi x^2}{r}(rh - xh)$

$$V = \pi x^2 h - \frac{\pi x^3 h}{r}$$

Differentiating with respect to x

$$\Rightarrow \frac{dV}{dx} = 2\pi xh - \frac{8\pi x^2 h}{r} \qquad \ldots (1)$$

for maximum volume $\frac{dV}{dx} = 0$

$$\Rightarrow 2\pi xh = \frac{8\pi x^2 h}{r}$$

$$\Rightarrow \ 2r = 3x \ \Rightarrow x = \frac{2}{3}r$$

Differentiating equation (1) with respect to x

$$\Rightarrow \frac{d^2y}{dx^2} = 2\pi h - \frac{6\pi xh}{r}$$

$$\left(\frac{d^2V}{dx^2}\right)_{x=\frac{2}{3}r} = 2\pi h - \frac{6\pi 2rh}{3r} = 2\pi h - 4\pi h = -2\pi h < 0$$

for maximum volume $x = \frac{2r}{3}$

$$y = \frac{rh - xh}{r} = \frac{rh - \frac{2}{3}rh}{r} = \frac{h}{3}$$

$\therefore$ Height of the right cylinder with maximum volume is $\frac{1}{3}rd$ height of cone.

$$Volume\ of\ the\ cone = \frac{1\pi r^2 h}{3}$$

Maximum volume of the cylinder $= \pi x^2 y = \frac{\pi 4r^2}{9}\frac{h}{3} = \frac{4\pi r^2 h}{27} = \frac{4}{9}\left(\frac{1}{3}\pi r^2 h\right)$

$$= \frac{4}{9}\ (Volume\ of\ the\ cone)$$

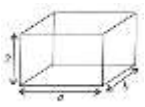
Hence proved.

12. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs Rs. 70 per square metre for the base and Rs. 45 per square metre for the sides, what is the cost of least expensive tank?

[CBSE 2019]

Solution

Let a m and b m be the length and breadth of rectangular tank respectively.



$\therefore$ Volume of tank $= 2ab = 9$ [Given]

$\Rightarrow \ ab = 4 \ \Rightarrow b = \frac{4}{a}$ ... (i)

If C is the total cost in rupees, then

$C = 70(ab) + 45(2a + 2b) \times 2$

$\Rightarrow \ C = 70ab + 90(2a + 2b)$

$\Rightarrow \ C = 70(a)\left(\frac{4}{a}\right) + 180\left(a + \frac{4}{a}\right)$ $[Using\ (i)]$

$\Rightarrow \ C = 280 + 180a + \frac{720}{a}$ ... (ii)

Differentiating (ii) w.r.t. 'a', we get

$\frac{dC}{da} = 180 - \frac{720}{a^2}$ and $\frac{d^2C}{da^2} = \frac{720 \times 2}{a^3}$

For maximum or minimum cost,

$\frac{dC}{da} = 0 \ \Rightarrow 180 - \frac{720}{a^2} = 0 \ \Rightarrow a^2 = 4 \ \Rightarrow a = 2$

For $a = 2, \frac{d^2C}{da^2} > 0 \ \Rightarrow C$ is least

Using (i), $a = 2$ and $b = 2$

Hence, cost of least expensive tank is

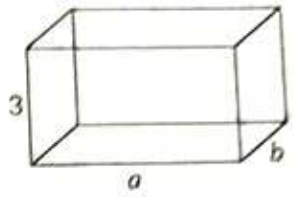
C = 280 + 360 + 360 = Rs. 1000

13. A tank with rectangular base and rectangular sides open at the top is to be constructed so that its depth is 3 m and volume is 75 m^3. If building of tank costs Rs. 100 per square metre for the base and Rs. 50 per square metre for the sides, find the cost of least expensive tank.

[CBSE 2015C]

Solution

Let a m and b m be the sides of the base of the tank.



$\therefore$ Volume of the tank $a.b.3 = 75\ m^3(given)$

$\Rightarrow ab = 25 \Rightarrow b = \frac{25}{a}$ ... (i)

If C is the total cost in rupees, then

$C = a \times b \times 100 + 2 \times 3 \times a \times 50 + 2 \times 3 \times b \times 50$

$= 100\ ab + 300(a + b) = 100 \times 25 + 300\left(a + \frac{25}{a}\right)$

$\Rightarrow \quad C = 2500 + 300\left(a + \frac{25}{a}\right)$

Differentiating w.r.t. a, we get

$\frac{dC}{da} = 300\left(1 - \frac{25}{a^2}\right)$ and

$\frac{d^2C}{da^2} = 300\left(0 + \frac{25 \times 2}{a^3}\right) = \frac{300 \times 50}{a^3}$

For maximum or minimum cost,

$\frac{dC}{da} = 0 \Rightarrow 1 - \frac{25}{a^2} = 0 \Rightarrow a = 5m$

and from (i) b = 5 m

$At\ a = 5; \frac{d^2C}{da^2} > 0 \Rightarrow C\ is\ minimum.$

Hence, the least cost of the tank is

$$C = \left[2500 + 300\left(5 + \frac{25}{5}\right)\right] = [2500 + 3000] = 5500.$$

14. Prove that the semi vertical angle of the right circular cone of given volume and least curved surface area is $\cos^{-1}\sqrt{2}$.

[CBSE 2014]

Solution

Let r, h, I, V and S be respectively the base radius, height slant height, volume and curved surface area of the cone. Then,

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$\boldsymbol{l^2 = r^2 + h^2, V = \frac{1}{3}\pi r^2 h}$ ... (i)

and $\mathbf{S = \pi r l = \pi r\sqrt{r^2 + h^2}}$

$\Rightarrow \quad S^2 = \pi^2 r^2 (r^2 + h^2) = \pi^2 \frac{3V}{\pi h}\left(\frac{3V}{\pi h} + h^2\right)$ $[Using\ (i)]$

$= 3\pi V\left(\frac{3V}{\pi h^2} + h\right)$

For S to be least, S^2 is also least.

$\therefore \frac{dS^2}{dh} = 3\pi V.\left(\frac{-6V}{\pi h^3} + 1\right)$ and

$\frac{d^2S^2}{dh^2} = 3\pi V\left(\frac{-6V}{\pi}\right).\frac{-3}{h^4} = \frac{54V^2}{h^4}$

For maximum or minimum S (and so S^2),

$\frac{dS^2}{dh} = 0 \Rightarrow 6V = \pi h^3$

$\Rightarrow \quad h = \left(\frac{6V}{\pi}\right)^{1/3}$ ... (ii)

For this value of h, $\frac{d^2S^2}{dh^2} = \frac{54V^2}{h^4} > 0$

$\Rightarrow \quad S^2$ and therefore S is least.

$\cot\theta = \frac{h}{r} = \frac{h}{\sqrt{\frac{3V}{\pi h}}} = \sqrt{\frac{\pi}{3V}}.h^{\frac{3}{2}}$ $[From\ (i)]$

$\Rightarrow \cot\theta = \sqrt{\frac{\pi}{3V}}\sqrt{\frac{6V}{\pi}} = \sqrt{2}$ $[From\ (ii)]$

$\Rightarrow$ The semi vertical angle, $\theta = \cot^{-1}\sqrt{2}$

15. Prove that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

[CBSE 2014, 2013]

Solution

Let r and h be the base radius and height of cylinder respectively.

$\therefore \quad \left(\frac{h}{2}\right)^2 + r^2 = R^2$ ... (i)

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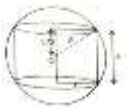
Now, V = Volume of the cylinder inscribed in a sphere

$= \pi r^2 h$



$\Rightarrow V = \pi h\left(R^2 - \frac{h^2}{4}\right)$ $[Using\ (i)]$

$\Rightarrow\ V = \pi\left(R^2 h - \frac{h^3}{4}\right)$

Now differentiating w.r.t. h, we get

$\frac{dV}{dh} = \pi\left(R^2 - \frac{3h^2}{4}\right)$ and $\frac{d^2V}{dh^2} = \pi\left(0 - \frac{3}{4}.2h\right)$

For maximum or minimum,

$\frac{dV}{dh} = 0 \Rightarrow R^2 - \frac{3}{4}h^2 = 0$

$\Rightarrow\ h^2 = \frac{4}{3}R^2 \Rightarrow h = \frac{2R}{\sqrt{3}}$

For this value of h, $\frac{d^2V}{dh^2} = -\frac{3}{2}\pi.\frac{2R}{\sqrt{3}} = -\sqrt{3}\pi R < 0$

$\Rightarrow$ V is maximum

Also maximum value of V

$$= \pi.\frac{2R}{\sqrt{3}}\left(R^2 - \frac{1}{4}.\frac{4}{3}R^2\right) = \pi.\frac{2R}{\sqrt{3}}.\frac{2}{3}R^2 = \frac{4\pi}{3\sqrt{3}}R^3 \text{ cu. units}$$

16. A window is of the form of a semi-circle with a rectangle on its diameter. The total perimeter of the window is 10 m. Find the dimension of the window to admit maximum light through the whole opening.

[CBSE 2014]

Solution

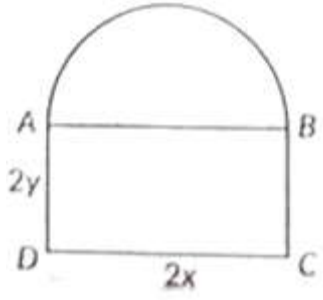
Let ABCD be a rectangle and let the semi-circle is described on the side AB as its diameter.

Let AB = 2x and AD = 2y. Let P = 10m be the given perimeter of window.

Therefore, 10 = 2x + 4y + πx

$\Rightarrow$ 4y = 10 – 2x – πx …(i)

Area of the window,



$A = (2x)(2y) + \frac{1}{2}\pi x^2$

$\Rightarrow \quad A = 4xy + \frac{1}{2}\pi x^2$

$\Rightarrow \quad A = 10x - 2x^2 - \pi x^2 + \frac{1}{2}\pi x^2 \qquad [using\ (i)]$

$\Rightarrow \quad A = 10x - 2x^2 - \frac{1}{2}\pi x^2$

On diff. w.r.t. x both sides, $\frac{dA}{dx} = 10 - 4x - \pi x$

Again diff. w.r.t.x both sides, $\frac{d^2A}{dx^2} = -(4 + \pi)$

For maxima or minima,

$\frac{dA}{dx} = 0 \Rightarrow 10 - 4x - \pi x = 0 \Rightarrow x = \frac{10}{4 + \pi}$

$\left[\frac{d^2A}{dx^2}\right]_{x=\frac{10}{4+\pi}} = -(4 + \pi) < 0$

So, A is maximum at $x = \left(\frac{10}{4+\pi}\right) m$.

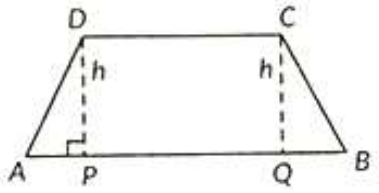
Now, length of the window is $2x = \left(\frac{20}{4+\pi}\right)$ m and width is $2y = \left(\frac{10}{4+\pi}\right)$ m.

17. If the length of three sides of a trapezium other than base is 10 cm each, then find the area of the trapezium when it is maximum.

[CBSE 2014C, 2013C]

Solution

Let ABCD be the given trapezium.



Then AD = DC = CB = 10 cm

In ΔAPD and ΔBQC

DP = CQ= h

AD = BC = 10 cm

∠DPA = ∠CQB = 90°

∴ $\Delta APD \cong \Delta BQC$ (by R.H.S. congruency)

⇒ AP = QB = x cm (Say)

∴ AB = AP + PQ + QB

= x + 10 + x = (2x + 10)cm

Also from ΔAPD, $AP^2 + PD^2 = AD^2$

⇒ $x^2 + h^2 = 10^2$

⇒ $h = \sqrt{100 - x^2}$ ...(i)

Now, area A of this trapezium is given by

$A = \frac{1}{2}(AB + DC).h = \frac{1}{2}(2x + 10 + 10)h$

$= (x + 10).\sqrt{100 - x^2}$ ...(ii) (Using (i))

Differentiating w.r.t. x, we get

$$\frac{dA}{dx} = 1.\sqrt{100 - x^2} + (x + 10).\frac{1}{2\sqrt{100 - x^2}}.(-2x)$$

$$= \frac{100 - x^2 - x^2 - 10x}{\sqrt{100 - x^2}} = \frac{-2(x^2 + 5x - 50)}{\sqrt{100 - x^2}}$$

$$= \frac{-2(x + 10)(x - 5)}{\sqrt{100 - x^2}}$$

For max. or min. value of A, $\frac{dA}{dx} = 0$

⇒ (x + 10)(x – 5) = 0 ⇒ x = 5 (Reject x = -10 as $\not< 0$)

For this value of x, $\frac{dA}{dx}$ changes sign from positive to dx negative.

∴ A is maximum at x = 5,

From (ii), the max. value of $A = (5 + 10).\sqrt{100 - 5^2}$

$$= 15\sqrt{75} = 75\sqrt{3} sq.cm.$$

18. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base.

[CBSE 2013C]

Solution

Let r be the base radius of the cone, I be the slant height and h be its height. Let V be its volume and S be its curved surface.

Now $V = \frac{1}{3}\pi r^2 h$

$\Rightarrow \quad h = \frac{3V}{\pi r^2}$

We have, $l^2 = h^2 + r^2$

Now S = **curved surface area of the cone = πrl** = $\pi r\sqrt{h^2 + r^2}$

S will be least $\Leftrightarrow$ S^2 will be least.

$\therefore \quad S^2 = \pi^2 r^2 (h^2 + r^2) = \pi^2 r^2 \left(\frac{9V^2}{\pi^2 r^4} + r^2\right)$

$\Rightarrow S^2 = \frac{9V^2}{r^2} + \pi^2 r^4$

Differentiating w.r.t.r, we get

$\frac{dS^2}{dr} = -\frac{18V^2}{r^3} + 4\pi^2 r^3$ and $\frac{d^2S^2}{dr^2} = \frac{54V^2}{r^4} + 12\pi^2 r^2$

For max. or min., $\frac{dS^2}{dr} = 0$

$\Rightarrow 4\pi^2 r^3 = \frac{18V^2}{r^3} = \frac{18}{r^3}\left(\frac{1}{3}\pi r^2 h\right)^2 = 2\pi^2 r h^2$

$\Rightarrow h^2 = 2r^2 \Rightarrow h = \sqrt{2}r \qquad (\because r \neq 0)$

For this value of r, $\frac{d^2S^2}{dr^2} > 0$

$\therefore$ Curved surface area of the cone is least when $h = \sqrt{2}r$.

19. If $f(x) = a(x - \cos x)$ is strictly decreasing in $\mathbb{R}$, then ' a ' belongs to
(a) [0]
(b) $(0, \infty)$
(c) $(-\infty, 0)$
(d) $(-\infty, \infty)$

[CBSE 2023]

Solution

(c): We have, $f(x) = a(x - \cos x)$
$\therefore f'(x) = a(1 + \sin x)$
It is given that $f(x)$ is a strictly decreasing function.
$\therefore f'(x) < 0 \Rightarrow a(1 + \sin x) < 0$
But$(1 + \sin x) \geq 0$ as $-1 \leq \sin x \leq 1$
$\therefore \ 1 + \sin x < 0$ is not possible.

$\Rightarrow a < 0 \therefore a \in (-\infty, 0)$

20. The interval in which the function $f(x) = 2x^3 + 9x^2 + 12x - 1$ is decreasing, is

(a) $(-1, \infty)$

(b) $(-2, -1)$

(c) $(-\infty, -2)$

(d) $[-1,1]$

[CBSE 2023]

Solution

(b): We have, $f(x) = 2x^3 + 9x^2 + 12x - 1$

$\Rightarrow f'(x) = 6x^2 + 18x + 12$

For decreasing, $f'(x) < 0$

$\therefore\ 6x^2 + 18x + 12 < 0 \Rightarrow x^2 + 3x + 2 < 0 \Rightarrow (x+1)(x+2) < 0 \Rightarrow -2 < x < -1$

So, $f(x)$ is decreasing, if $x \in (-2, -1)$.

21. Show that the function $f(x) = \frac{16\sin x}{4+\cos x} - x$, is strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$.

[CBSE 2023]

Solution

Given, $f(x) = \frac{16\sin x}{4+\cos x} - x$

$$\therefore f'(x) = \frac{(4+\cos x)(16\cos x) - (16\sin x)(-\sin x)}{(4+\cos x)^2} - 1 = \frac{64\cos x + 16\cos^2 x + 16\sin^2 x}{(4+\cos x)^2} - 1$$

$$= \frac{64\cos x + 16(\cos^2 x + \sin^2 x)}{(4+\cos x)^2} - 1 = \frac{64\cos x + 16}{(4+\cos x)^2} - 1 (\because \cos^2 x + \sin^2 x = 1)$$

$$= \frac{64\cos x + 16 - (4+\cos x)^2}{(4+\cos x)^2} = \frac{64\cos x + 16 - 16 - \cos^2 x - 8\cos x}{(4+\cos x)^2}$$

$$= \frac{56\cos x - \cos^2 x}{(4+\cos x)^2}$$

A function $f(x)$ is strictly decreasing in interval l if $f'(x) < 0$ in I.

$$\Rightarrow \frac{56\cos x - \cos^2 x}{(4+\cos x)^2} < 0$$

As $(4+\cos x)^2 > 0$

$\Rightarrow 56\cos x - \cos^2 x < 0 \Rightarrow \cos x(56 - \cos x) < 0$

$\Rightarrow \cos x < 0 (\because \cos x > 56$ is not possible as $-1 \le \cos x \le 1)$

Now, in the interval $\left(\frac{\pi}{2}, \pi\right), \cos x < 0$

$\therefore f(x)$ is strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$.

22. The median of an equilateral triangle is increasing at the rate of $2\sqrt{3}$ cm/s. Find the rate at which its side is increasing.

[CBSE 2023]

Solution

. Let the side of an equilateral triangle be $2x$ cm, then its median $= \sqrt{3}x$

Let $M = \sqrt{3}x \Rightarrow \frac{dM}{dt} = \sqrt{3}\frac{dx}{dt}$

$$\Rightarrow 2\sqrt{3} = \sqrt{3}\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{2\sqrt{3}}{\sqrt{3}} = 2 \text{ cm/sec}$$

Now, side of triangle $= 2x$

$$\Rightarrow s = 2x \Rightarrow \frac{d\,s}{dt} = 2\frac{dx}{dt} = 2 \times 2 \text{ cm/sec} = 4 \text{ cm/sec}$$

So, side is increasing at the rate of 4 cm/sec.

23. Given a curve $y = 7x - x^3$ and x increases at the rate of 2 units per second. The rate at which the slope of the curve is changing, when $x = 5$ is

(a) -60 units/sec

(b) 60 units /sec

(c) -70 units/sec

(d) -140 units/sec

[CBSE 2024]

Solution

(a): Given curve is $y = 7x - x^3$.

24. Differentiating both sides w.r.t. x, we get

$\frac{dy}{dx} = 7 - 3x^2$

Let m be the slope, then $m = 7 - 3x^2$

Differentiating both sides w.r.t. t, we get

$\Rightarrow \frac{dm}{dt} = -\frac{6xdx}{dt}$

It is given that $\frac{dx}{dt} = 2$ units /sec and $x = 5$

$\Rightarrow \frac{dm}{dt} = -6(5)(2) = -60$ units /sec

The surface area of a cube increases at the rate of 72 cm^2/sec. Find the rate of change of its volume, when the edge of the cube measures 3 cm .

[CBSE 2024]

Solution

Let a be the edge of the cube.

Surface area of cube, $s = 6a^2$

Now, $\frac{ds}{dt} = 12a\frac{da}{dt} \Rightarrow 12a\frac{da}{dt} = 72$ [Given]

$\Rightarrow \frac{da}{dt} = \frac{6}{a}$

Volume of the cube is $V = a^3$

$\Rightarrow \frac{dV}{dt} = 3a^2\frac{da}{dt} = 3(3)^2 \times \frac{6}{3} = 54$ cm^3/s

Therefore, when the edge of the cube is 3 cm , then the rate at which the cube's volume is changing is 54 cm^3/s.

25. Show that $f(x) = e^x - e^{-x} + x - \tan^{-1} x$ is strictly increasing in its domain.

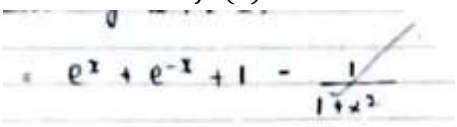
[CBSE 2024]

Solution

$f(x) = e^{+x} - e^{-x} + x - \tan^{-1} x$

Different lating w.rt x,

$f'(x)$



$= e^x + e^{-x} + \frac{1 + x^2 - 1}{1 + x^2}$

$= e^x + e^{-x} + \frac{x^2}{1 + x^2}$

For ang value of x in its demain,

$$e^{x^x} > 0, e^{-x} > 0, \frac{x^2}{1+x^2} > 0$$

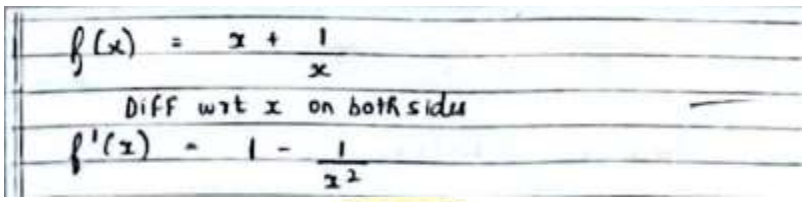
26. So, $f'(x) = e^x + e^{-x} + \frac{x^2}{1+x^2} > 0$

As $f'(x)$ is always greater than $0, f(x)$ is a stridly increasing function

If M and m denote the local maximum and local minimum values of the function $f(x) = x + \frac{1}{x}(x \neq 0)$ respectively, find the value of $(M - m)$.

[CBSE 2024]

Solution



Put $f'(x) = 0$

$$1 - \frac{1}{x^2} = 0 \Rightarrow 1 - \frac{1}{x^2} \Rightarrow x^2 = 1$$

So, $x = \pm 1$.

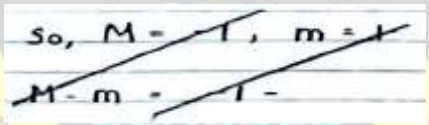
Now, $f''(x) = \frac{2}{x^3}$

$f''(1) = \frac{2}{1} > 0$.

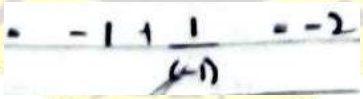
$f(x)$ attains minima at $x = 1$.

$f''(-1) = \frac{2}{-1} < 0$

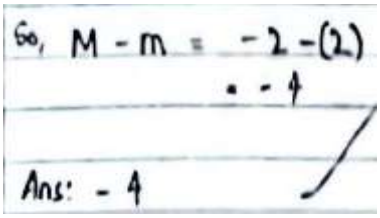
$f(x)$ attains maxima at $x = -1$.



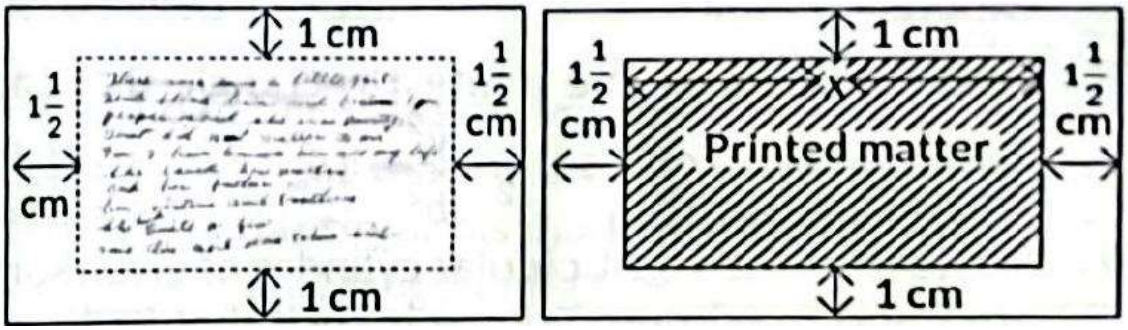
Local maximum value $= M = F(-1)$



Local minimum value $= m = f(1) =$



27. Case Study : A rectangular visiting card is to contain 24 sq. cm of printed matter. The margins at the top and bottom of the card are to be 1 cm and the margins on the left and right are to be $1\frac{1}{2}$ cm as shown below:



On the basis of the above information, answer the following questions :

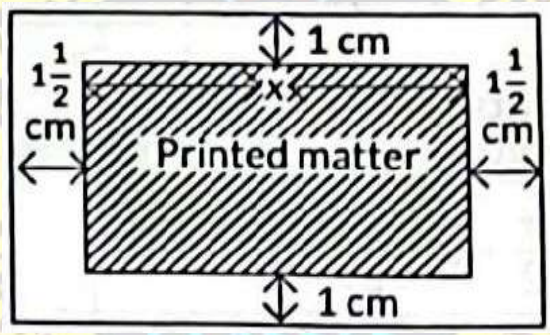
(i) Write the expression for the area of the visiting card in terms of x.

(ii) Obtain the dimensions of the card of minimum area.

[CBSE 2024]

Solution

Let breadth of card be cm.



Given, area of printing matter $= 24\text{sq. cm}$

$\Rightarrow\ xy = 24$

$$\Rightarrow y = \frac{24}{x} \#(i)$$

$\therefore$ Length of rectangular visiting card $= x + 2\left(\frac{3}{2}\right) = (x+3)\text{cm}$ and breadth of rectangular visiting card $= \gamma + 2(1) = (y+2)\text{cm}$

(i) Area of visiting card $(A) = (x+3)(y+2)$

$$= (x+3)\left(\frac{24}{x} + 2\right)$$

$$\Rightarrow A = 30 + 2x + \frac{72}{x},$$

(Using (i))

which is the required expression.

(ii) $\frac{dA}{dx} = 2 - \frac{72}{x^2}$ and $\frac{d^2A}{dx^2} = \frac{144}{x^3}$

For minimum area, put $\frac{dA}{dx} = 0 \Rightarrow 2 - \frac{72}{x^2} = 0$

$\Rightarrow x^2 = 36 \Rightarrow x = \pm 6$

At $x = 6, \frac{d^2A}{dx^2} = \frac{144}{(6)^3} > 0$

So, A has minimum value at $x = 6$.

$\therefore\ y = \frac{24}{6} = 4$ (Using(i))

$\therefore$ Dimensions of rectangular visiting card are $(6 + 3)$cm and $(4 + 2)$cm i.e., 9 cm and 6 cm .

28. A spherical medicine ball when dropped in water dissolves in such a way that the rate of decrease of volume at any instant is proportional to its surface area. Calculate the rate of decrease of its radius.

[CBSE 2025]

Solution

. Let ' V be the volume of the spherical medicine ball, ' r ' be the radius of the spherical ball, ' t ' be the time and ' K ' be the constant of proportionality.

$\therefore$ Volume of sphere $= \frac{4}{3}\pi r^3$

Surface area of sphere $= 4\pi r^2$

Given, $\frac{dV}{dt} = -KA\ \ \therefore\ \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = -K(4\pi r^2)$

$$\Rightarrow \frac{4}{3}\pi \times 3r^2 \times \frac{dr}{dt} = -K(4\pi r^2) \Rightarrow \frac{dr}{dt} = -K$$

This shows that the rate of decrease of its radius is constant.

29. The area of an expanding rectangle is increasing at the rate of 48 cm^2/s. The length of the rectangle is always square of its breadth. At what rate the length of rectangle increasing at an instant, when breadth = 4.5 cm ?

[CBSE 2025]

Solution

. Given, $\frac{dA}{dt} = 48\ \text{cm}^2/\text{s}$

Area of rectangle $A = Ib = \beta^{3/2}$

$[\because 1 = b^2]$

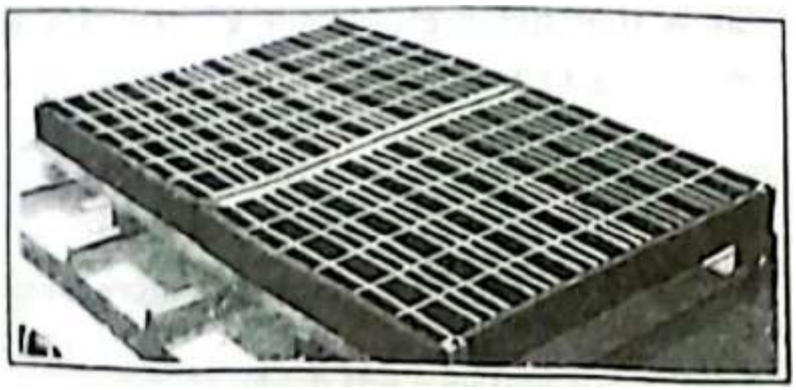
Differentiating w.r.t. t, we get $\frac{dA}{dt} = \frac{3}{2} I^{1/2} \cdot \frac{dl}{dt}$

$$\Rightarrow 48 = \frac{3}{2} l^{1/2} \cdot \frac{dl}{dt} \Rightarrow \frac{32}{\sqrt{1}} = \frac{dl}{dt}$$

$\Rightarrow \frac{dl}{dt} = \frac{32}{4.5} = \frac{32\times2}{9} = \frac{64}{9}$ cm/s$[\because b = 4.5$ cm$]$

30. Case Study : A technical company is designing a rectangular solar panel installation on a roof using 300 metres of boundary material. The design includes a partition running parallel to one of the sides dividing the area (roof) into two sections.

Let the length of the side perpendicular to the partition be x metres and with parallel to the partition be y metres.



Based on this information, answer the following questions:

(i) Write the equation for the total boundary material used in the boundary and parallel to the partition in terms of x and y.

(ii) Write the area of the solar panel as a function of x.

(iii) (a) Find the critical points of the area function. Use second derivative test to determine critical points at the maximum area. Also, find the maximum area.

OR

(iii) (b) Using first derivative test, calculate the maximum area the company can enclose with the 300 metres of boundary material, considering the parallel partition

Solution

. Given, the total length of the boundary material is 300 metres.

The length of side perpendicular to the partition is x metres.

The length of side parallel to the partition is y metres.

(i) The total length of the boundary material is

$$2x + 3y = 300 \#(i)$$

(ii) Area of the solar panel is, $A = xy$

Since, $2x + 3y = 300$

$\Rightarrow 3y = 300 - 2x \Rightarrow y = \frac{300-2x}{3}$

Put y in equation (i), we get

$\Rightarrow A(x) = x\left[\frac{300-2x}{3}\right] = 100x - \frac{2}{3}x^2$

(iii) (a) We have, $A(x) = 100x - \frac{2}{3}x^2$

Differentiate w.r.t. x, we get

$\Rightarrow A'(x) = 100 - \frac{4}{3}x$

For critical point, $A'(x) = 0$

$\Rightarrow 100 - \frac{4}{3}x = 0 \Rightarrow x = \frac{300}{4} = 75$

Again, $A''(x) = \frac{-4}{3}$

Now, $A''(x)|_{x=75} = \frac{-4}{3} < 0$, i.e., $x = 75$ is a point of maxima.

$\therefore$ Maximum area $= A(75) = 100(75) - \frac{2}{3}(75)^2$

$$= 7500 - 3750 = 3750 \text{ m}^2 \#(i)$$

OR

(iii) (b) We have, $A = xy$

Also, perimeter $= 2x + 3y = 300 \Rightarrow 2x = 300 - 3y$

$\Rightarrow x = \left(150 - \frac{3y}{2}\right)$

Put the value of x in equation (i), we get

$\therefore A(y) = \left(150 - \frac{3y}{2}\right) y \; A(y) = 150y - \frac{3y^2}{2}$

$\therefore \; A'(y) = 150 - 3y$

For critical points, $\therefore 150 - 3y = 0$

$\Rightarrow 3y = 150 \Rightarrow y = 50$

Since, $A'(y)$ changes sign from positive to negative at $y = 50$. This point is a point of maxima.

$\therefore A(50) = \left(150 - \frac{3 \times 50}{2}\right) \times 50$

[From (i)]

$= 3750 \text{ m}^2$

31. .Case Study : A ladder of fixed length ' h ' is to be placed along the wall such that it is free to move along the height of the wall.

Based upon the above information, answer the following questions:

(i) Express the distance (y) between the wall and foot of the ladder in terms ' h ' and height (x) on the wall at a certain instant. Also, write an expression in terms of h and x for the area (A) of the right triangle, as seen from the side by an observer.

(ii) Find the derivative of the area (A) with respect to the height on the wall (x), and find its critical point.

(iii) (a) Show that the area (A) of the right triangle is maximum at the critical point.

OR

(iii) (b) If the foot of the ladder whose length is 5 m , is being pulled towards the wall such that rate of decrease of distance (y) is 2 m/s, then at what rate is the height on the wall (x) increasing, when the foot of the foot of the ladder is 3 m away from the wall?

[CBSE 2025]

Solution

(i) Using pythagoras thereom, $h^2 = x^2 + y^2$ or $y = \sqrt{h^2 - x^2}$

Area of $\triangle ABC = \frac{1}{2} \times x \times y$

$= \frac{1}{2} \times x \times \sqrt{h^2 - x^2}$

(ii) $\frac{d\,A}{dx} = \frac{1}{2}\sqrt{h^2 - x^2} + \frac{1}{2} \times x \times \frac{1}{2\sqrt{h^2 - x^2}}(-2x)$

$= \frac{1}{2}\sqrt{h^2 - x^2} - \frac{x^2}{2\sqrt{h^2 - x^2}} = \frac{h^2 - 2x^2}{2\sqrt{h^2 - x^2}}$

For critical point; $\frac{dA}{dx} = 0$

$\Rightarrow h^2 - 2x^2 = 0$

$\Rightarrow (h - \sqrt{2}x)(h + \sqrt{2}x) = 0$

$\Rightarrow x = \frac{h}{\sqrt{2}}$ $\left[\because x \neq \frac{-h}{\sqrt{2}}\right]$

(iii) (a) $\frac{d^2A}{dx^2} = \frac{1}{2}\left[\frac{\sqrt{h^2-x^2}(-4x) - (h^2-2x^2) \times \frac{(-2x)}{2\sqrt{h^2-x^2}}}{h^2-x^2}\right]$

$= \frac{(h^2-x^2)(-4x) + (h^2-2x^2)x}{2(h^2-x^2)\sqrt{h^2-x^2}}$

At $x = \frac{h}{\sqrt{2}}, \frac{d^2A}{dx^2} = \frac{\left(\frac{h^2}{2}\right)\left(\frac{-4h}{\sqrt{2}}\right) + 0}{2\left(\frac{h^2}{2}\right) \times \frac{h}{\sqrt{2}}} = -2 < 0$

$\therefore$ Area is maximum at $x = \frac{h}{\sqrt{2}}$.

OR

(iii) (b) We have, $\frac{dy}{dt} = -2$ m/s

Using Pythagoras theorem, $x^2 + y^2 = 25$

If $y = 3$, then $x = \sqrt{25 - 9} = \sqrt{16} = 4$ cm

Differentiate (i) w.r.t. t, we get

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$\Rightarrow 2 \times \frac{4dx}{dt} + 2 \times 3 \times (-2) = 0$

$\Rightarrow \frac{dx}{dt} = \frac{3}{2}$ m/s

$\therefore$ The height of the wall is increasing at the rate of $\frac{3}{2}$ m/s.

32. Find the absolute maximum and absolute minimum of function $f(x) = 2x^3 - 15x^2 + 36x + 1$ on [1,5].

[CBSE 2025]

Solution

Given, $f(x) = 2x^3 - 15x^2 + 36 + 1$

Differentiating on both sides, w.r.t. x, we get

$f'(x) = 6x^2 - 30x + 36$

For critical points, $f'(x) = 0$

$\Rightarrow 6x^2 - 30x + 36 = 0 \Rightarrow x^2 - 5x + 6 = 0 \Rightarrow (x-2)(x-3) = 0 \Rightarrow x = 2,3$

We have been given interval [1,5]

Hence, calculating $f(x)$ at 1,2,3 and 5

$f(1) = 2(1)^3 - 15(1)^2 + 36(1) + 1 = 2 - 15 + 36 + 1 = 24 f(2)$
$= 2(2)^3 - 15(2)^2 + 36(2) + 1 = 16 - 60 + 72 + 1 = 29 f(3)$
$= 2(3)^3 - 15(3)^2 + 36(3) + 1 = 54 - 135 + 108 + 1 = 28 f(5)$
$= 2(5)^3 - 15(5)^2 + 36(5) + 1 = 250 - 375 + 180 + 1 = 56$

Hence, absolute maximum value is 56 at $x = 5$ and absolute minimum value is 24 at $x = 1$.

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Integrals

1. Find : $\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx$

[CBSE 2017]

Solution

We have, $\int \frac{\sin^2 x - \cos^2 c}{\sin x \cos x} dx$

$$= \int \frac{\sin^2 x}{\sin x \cos x} dx - \int \frac{\cos^2 x}{\sin x \cos x} dx$$

$$= \int \tan x dx - \int \cot x dx$$

$$= \ln|\sec x| - \ln|\sin x| + C = \ln\left|\frac{1}{\sin x \cos x}\right| + C$$

$$= \ln\left|\frac{2}{2\sin x \cos x}\right| + C = \ln|2\operatorname{cosec} 2x| + C$$

2. Evaluate: $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx$

[CBSE 2018]

Solution

Let $I = \int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx$

$$= \int \frac{\cos^2 x - \sin^2 x + 2\sin^2 x}{\cos^2 x} dx$$

$$= \int \frac{\cos^2 x + \sin^2 x}{\cos^2 x} dx = \int \frac{1}{\cos^2 x} dx = \int \sec^2 x dx$$

$$= \tan x + C$$

3. Integrate the function $\frac{\cos(x+a)}{\sin(x+b)}$ with respect to x.

[CBSE 2019]

Solution

We have, $\int \frac{\cos(x+a)}{\sin(x+b)} dx = \int \frac{\cos(x+b+a-b)}{\sin(x+b)} dx$

$$= \int \frac{\cos(x+b)\cos(a-b) - \sin(x+b)\sin(a-b)}{\sin(x+b)} dx$$

$$= \cos(a-b)\int \frac{\cos(x+b)}{\sin(x+b)} dx - \sin(a-b)\int dx$$

$$= \cos(a-b)\log\sin(x+b) - x\sin(a-b) + C$$

4. Evaluate: $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} dx$

[CBSE 2013]

Solution

5. Let $I = \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} dx$

$= \int \frac{(2\cos^2 x - 1) - (2\cos^2 \alpha - 1)}{\cos x - \cos \alpha} dx$

$= 2\int \frac{\cos^2 x - \cos^2 \alpha}{\cos x - \cos \alpha} dx$

$= 2\int \frac{(\cos x + \cos \alpha) \cdot (\cos x - \cos \alpha)}{\cos x - \cos \alpha} dx$

$= 2\int (\cos x + \cos \alpha) dx = 2[\sin x + x\cos \alpha] + C$

Evaluate: $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x . \cos^2 x} dx$

[CBSE 2014]

Solution

Let $I = \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cdot \cos^2 x} dx$

$= \int \frac{\sin^6 x}{\sin^2 x \cdot \cos^2 x} dx + \int \frac{\cos^6 x}{\sin^2 x \cdot \cos^2 x} dx$

$= \int \frac{\sin^4 x}{\cos^2 x} dx + \int \frac{\cos^4 x}{\sin^2 x} dx$

$= \int \frac{\sin^2 x(1 - \cos^2 x)}{\cos^2 x} dx + \int \frac{\cos^2 x(1 - \sin^2 x)}{\sin^2 x} dx$

$= \int \tan^2 x dx - \int \sin^2 x dx + \int \cot^2 x dx - \int \cos^2 x dx$

$= \int (\sec^2 x - 1) dx - \int \sin^2 x dx + \int (\text{cosec}^2 x - 1) dx - \int (1 - \sin^2 x) dx$

$= \tan x - x + (-\cot x) - x - x + C = \tan x - \cot x - 3x + C$

6. Find $\int \frac{\sin 2x}{\sqrt{9 - \cos^4 x}} dx$.

[CBSE SQP 2020-21]

Solution

Let $I = \int \frac{\sin 2x}{\sqrt{9 - \cos^4 x}} dx$

put $\cos^2 x = t$

$\Rightarrow -2\cos x \sin x dx = dt \Rightarrow \sin 2x dx = -dt$

$\therefore I = -\int \frac{dt}{\sqrt{3^2 - t^2}} = -\sin^{-1} \frac{t}{3} + c = -\sin^{-1} \left(\frac{\cos^2 x}{3} \right) + c$

7. Find: $\int \frac{\sin^6 x}{\cos^8 x} dx$

[CBSE 2020]

Solution

Let $I = \int \frac{\sin^6 x}{\cos^8 x} dx = \int \frac{\sin^6 x}{\cos^6 x \cdot \cos^2 x} dx$

$= \int \tan^6 x \sec^2 x dx$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$

$\therefore\ I = \int t^6 dt = \frac{t^7}{7} + C = \frac{1}{7}\tan^7 x + C$

Type- Quadratic Formula Based

8. Find : $\int \frac{dx}{\sqrt{4x-x^2}}$

[CBSE 2021-22]

Solution

$\int \frac{dx}{\sqrt{4x - x^2}} = \int \frac{dx}{\sqrt{4 - (x-2)^2}}$

Put $x - 2 = 2\sin\theta$

$\Rightarrow dx = 2\cos\theta d\theta$

$= \int \frac{2\cos\theta}{\sqrt{4 - (2\sin\theta)^2}} d\theta$

Now, $4 - (2\sin\theta)^2 = 4 - 4\sin^2\theta = 4(1 - \sin^2\theta) = 4\cos^2\theta$

$= \int \frac{2\cos\theta d\theta}{\sqrt{4\cos^2\theta}} = \int \frac{2\cos\theta d\theta}{2\cos\theta} = \theta + c$

$= \sin^{-1}\left(\frac{x-2}{2}\right) + c,\ \left[\text{From}(i), \theta = \sin^{-1}\left(\frac{x-2}{2}\right)\right]$

where c is an arbitrary constant.

9. Find : $\int \frac{dx}{\sqrt{5-4x-2x^2}}$

[CBSE 2019]

Solution

Let $I = \int \frac{dx}{\sqrt{5-4x-2x^2}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{5}{2} - 2x - x^2}}$

$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{7}{2} - 1 - 2x - x^2}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(\sqrt{\frac{7}{2}}\right)^2 - (x+1)^2}}$

$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{x+1}{\sqrt{\frac{7}{2}}}\right) + C = \frac{1}{\sqrt{2}} \sin^{-1}\left[\sqrt{\frac{2}{7}}(x+1)\right] + C$

10.Find $\int \frac{x+3}{\sqrt{5-4x-2x^2}} dx$.

[CBSE 2015C]

Solution

Let $I = \int \frac{x+3}{\sqrt{5-4x-2x^2}}dx$

Let $x + 3 = A\frac{d}{dx}(5 - 4x - 2x^2) + B = A(-4 - 4x) + B$

On comparing the coefficients of like term, we get

$-4A = 1 \Rightarrow A = -\frac{1}{4}$ and $-4A + B = 3 \Rightarrow B = 2$

$$x + 3 = -\frac{1}{4}(-4 - 4x) + 2$$

$$\Rightarrow I = \int \frac{-\frac{1}{4}(-4 - 4x) + 2}{\sqrt{5 - 4x - 2x^2}}dx$$

$$= \frac{-1}{4}\int \frac{-4 - 4x}{\sqrt{5 - 4x - 2x^2}}dx + 2 \cdot \int \frac{1}{\sqrt{5 - 4x - 2x^2}}dx$$

$$\Rightarrow -\frac{1}{4}I_1 + 2I_2$$

(i)

where $I_1 = \int \frac{-4-4x}{\sqrt{5-4x-2x^2}}dx$

Put $5 - 4x - 2x^2 = t \Rightarrow (-4 - 4x)dx = dt$

$$\therefore\ I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}dt = 2\sqrt{t} + C_1$$

$$= 2\sqrt{5 - 4x - 2x^2} + C_1$$

and $I_2 = \int \frac{dx}{\sqrt{5-4x-2x^2}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\frac{5}{2}-2x-x^2}}$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\left(\sqrt{\frac{7}{2}}\right)^2 - (x+1)^2}} = \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x+1}{\sqrt{7/2}}\right) + C_2$$

From (i), (ii) and (iii), we get

$$I = -\frac{1}{4} \cdot 2\sqrt{5 - 4x - 2x^2} + 2 \cdot \frac{1}{\sqrt{2}}\sin^{-1}\left[\sqrt{\frac{2}{7}}(x+1)\right] + C$$

where $C = C_1 + C_2$

$$\Rightarrow I = -\frac{1}{2}\sqrt{5 - 4x - 2x^2} + \sqrt{2}\sin^{-1}\left[\sqrt{\frac{2}{7}}(x+1)\right] + C$$

10. Find: $\int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)}dx$

[CBSE 2018]

Solution

Let $I = \int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)}dx$

Put $\sin x = t \Rightarrow \cos x dx = dt$

$\therefore I = 2\int \frac{1}{(1-t)(1+t^2)}dt$

Now, $\frac{1}{(1-t)(1+t^2)} = \frac{A}{(1-t)} + \frac{Bt+C}{(1+t^2)}$

$\Rightarrow 1 = A(1+t^2) + (Bt+C)(1-t)$

$\Rightarrow 1 = A + At^2 + Bt - Bt^2 + C - Ct$

$\Rightarrow 1 = (A+C) + t^2(A-B) + t(B-C)$

On equating coefficients of like terms on both sides, we get

$A + C = 1, A - B = 0 \Rightarrow A = B$

$B - C = 0 \Rightarrow B = C$

$\therefore\ A = B = C$

$\therefore A = \frac{1}{2}, B = \frac{1}{2}, C = \frac{1}{2}$

$\therefore \frac{1}{(1-t)(1+t^2)} = \frac{1}{2(1-t)} + \frac{1}{2}\frac{t}{(1+t^2)} + \frac{1}{2(1+t^2)}$

Put above equation in (i), we get

$I = 2\left[\int \frac{1}{2(1-t)}dt + \int \frac{1}{2}\frac{t}{(1+t^2)}dt + \int \frac{1}{2(1+t^2)}dt\right]$

$= \int \frac{dt}{(1-t)} + \int \frac{t}{(1+t^2)}dt + \int \frac{dt}{(1+t^2)}$

$= -\log(1-t) + \frac{1}{2}\log(1+t^2) + \tan^{-1}(t) + C$

$= -\text{iog}(1-\sin x) + \frac{1}{2}\log(1+\sin^2 x) + \tan^{-1}(\sin x) + C$

$= -\log(1-\sin x) + \log(1+\sin^2 x)^{1/2} + \tan^{-1}(\sin x) + C$

$= \log\frac{\sqrt{1+\sin^2 x}}{1-\sin x} + \tan^{-1}(\sin x) + C$

11. Find: $\int \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}dx$

[CBSE 2016]

Solution

Let $I = \int \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}dx$

Let $x^2 = t$

$\therefore \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)} = \frac{(t+1)(t+4)}{(t+3)(t-5)}$

$= \frac{t^2+5t+4}{(t+3)(t-5)} = 1 + \frac{7t+19}{(t+3)(t-5)}$

Let $\frac{7t+19}{(t+3)(t-5)} = \frac{A}{t+3} + \frac{B}{t-5}$

$\Rightarrow 7t + 19 = A(t-5) + B(t+3)$

Putting $t = 5$, we get $B = \frac{27}{4}$

Putting $t = -3$, we get $A = \frac{1}{4}$

$$\therefore \ \frac{t^2 + 5t + 4}{(t+3)(t-5)} = 1 + \frac{1}{4(t+3)} + \frac{27}{4(t-5)}$$

$$\Rightarrow I = \int \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}dx = \int dx + \frac{1}{4}\int \frac{1}{(x^2+3)}dx + \frac{27}{4}\int \frac{1}{(x^2-5)}dx$$

$$= x + \frac{1}{4\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + \frac{27}{4} \times \frac{1}{2\sqrt{5}}\log\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right| + C$$

$$= x + \frac{1}{4\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + \frac{27}{8\sqrt{5}}\log\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right| + C$$

12. Evaluate: $\int \frac{dx}{x(x^5+3)}$

[CBSE 2013]

Solution

Let $I = \int \frac{dx}{x(x^5+3)}$

Put $x^5 + 3 = t \Rightarrow 5x^4 dx = dt$

$$\Rightarrow dx = \frac{dt}{5x^4}$$

$$\therefore \ I = \frac{1}{5}\int \frac{dt}{t(t-3)} = \frac{1}{5}\int \left[\frac{1}{3(t-3)} - \frac{1}{3t}\right]dt$$

$$= \frac{1}{15}[\log|t-3| - \log|t|] + C = \frac{1}{15}\log\left|\frac{t-3}{t}\right| + C$$

$$= \frac{1}{15}\log\left|\frac{x^5}{x^5+3}\right| + C$$

13. Find: $\int e^x \cdot \sin 2x \, dx$

[CBSE 2021-22]

Solution

$\int e^x \cdot \sin 2x dx$

Let $I = \int e^x \sin 2x dx$

$= \left[\sin 2x e^x - 2\int \cos 2x e^x dx\right]$

$= \left[e^x \sin 2x - 2\int e^x \cos 2x dx\right]$

$= \left[e^x \sin 2x - 2\left[\cos 2x e^x + 2\int \sin 2x e^x dx\right]\right]$

$I = e^x \sin 2x - 2e^x \cos 2x - 4I + C$

(From (i))

$$\Rightarrow 51 = e^x \sin 2x - 2e^x \cos 2x + \frac{C}{5}$$

$\Rightarrow\ I = \frac{1}{5}(e^x \sin 2x - 2e^x \cos 2x) + \frac{c}{5}$

$= \frac{e^x}{5}(\sin 2x - 2\cos 2x) + C_1$, where $C_1 = \frac{C}{5}$ is an arbitrary constant.

14. Find: $\int \sec^3 x\, dx$

[CBSE 2020]

Solution

Let $I = \int \sec^3 x dx = \int \sec x \cdot \sec^2 x dx$

$= \sec x \tan x - \int \sec x \tan x \cdot \tan x dx$

(Applying integration by parts)

$= \sec x \tan x - \int \sec x(\sec^2 x - 1)dx$

$= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx$

$= \sec x \tan x - 1 + \ln|\sec x + \tan x| + C_1,$

$\Rightarrow\ 21 = \sec x \tan x + \ln|\sec x + \tan x| + C_1$

$\therefore\ I = \frac{1}{2}(\sec x \tan x) + \frac{1}{2}\ln|\sec x + \tan x| + C$

15. Find: $\int \frac{(x^2+1)e^x}{(x+1)^2} dx$

[CBSE 2015C]

Solution

Let $I = \int \frac{x^2+1}{(x+1)^2} e^x dx$

$= \int e^x \cdot \left[\frac{(x^2-1)+2}{(x+1)^2}\right] dx$

$= \int e^x \cdot \left[\frac{x^2-1}{(x+1)^2} + \frac{2}{(x+1)^2}\right] dx$

$= \int e^x \cdot \left[\frac{x-1}{x+1} + \frac{2}{(x+1)^2}\right] dx$

Take $f(x) = \frac{x-1}{x+1}$

$\Rightarrow f'(x) = \frac{(x+1)\cdot 1 - (x-1)\cdot 1}{(x+1)^2} = \frac{2}{(x+1)^2}$

By using the formula, we get

$I = \int e^x[f(x) + f'(x)]dx = e^x f(x) + C$

$I = e^x \cdot \left[\frac{x-1}{x+1}\right] + C$

16. Evaluate: $\int \frac{x \cos^{-1} x}{\sqrt{1-x^2}} dx$

[CBSE 2014]

Solution

Let $I = \int \frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx$

Put $\cos^{-1}x = \theta \Rightarrow x = \cos\theta \Rightarrow dx = -\sin\theta d\theta$

$\Rightarrow I = \int \frac{\cos\theta(\theta)}{\sqrt{1-\cos^2\theta}}(-\sin\theta)d\theta$

$\Rightarrow I = -\int \theta\cos\theta d\theta$

$\Rightarrow \ -I = \theta\int \cos\theta d\theta - \int \left(\frac{d}{d\theta}\theta\int \cos\theta d\theta\right)d\theta$

(Applying integration by parts)

$\Rightarrow \ -I = \theta\sin\theta - \int \sin\theta d\theta$

$\Rightarrow -1 = \theta\sin\theta + \cos\theta + C$

$\Rightarrow I = -\left[\theta\sqrt{1-\cos^2\theta} + \theta\right] + C$

$\Rightarrow \ I = -\left[\sqrt{1-x^2}\cos^{-1}x + x\right] + C$

17. Find $\int \frac{(3\sin\theta-2)\cos\theta}{5-\cos^2\theta-4\sin\theta}d\theta$.

[CBSE 2016, 2013C]

Solution

Let $I = \int \frac{(3\sin\theta-2)\cos\theta}{5-\cos^2\theta-4\sin\theta}d\theta$

(Put $\cos^2\theta = 1-\sin^2\theta$)

$= 3\int \frac{\sin\theta\cos\theta}{4+\sin^2\theta-4\sin\theta}d\theta - 2\int \frac{\cos\theta}{4+\sin^2\theta-4\sin\theta}d\theta$

$= 3I_1 - 2I_2$ (say)

Now, $I_1 = \int \frac{\sin\theta\cos\theta}{4+\sin^2\theta-4\sin\theta}d\theta$

Put $\sin^2\theta = t \Rightarrow 2\sin\theta\cos\theta d\theta = dt$

$\therefore \ I_1 = \frac{1}{2}\int \frac{dt}{4+t-4\sqrt{t}} = \frac{1}{2}\int \frac{dt}{(\sqrt{t}-2)^2}$

Put $\sqrt{t}-2 = u \Rightarrow \sqrt{t} = u+2$

$\Rightarrow \frac{1}{2\sqrt{t}}dt = du \Rightarrow dt = 2(u+2)du$

$\therefore \ I_1 = \int \frac{(u+2)}{u^2}du = \int \frac{du}{u} + 2\int \frac{du}{u^2}$

$= \log u - \frac{2}{u} + C_1 = \log(\sqrt{t}-2) - \frac{2}{\sqrt{t}-2} + C_1$

$= \log(\sin\theta - 2) - \dfrac{2}{\sin\theta - 2} + C_1$

Also, $I_2 = \int \frac{\cos\theta}{4+\sin^2\theta-4\sin\theta} d\theta$

Put $\sin\theta = m \Rightarrow \cos\theta d\theta = dm$

$\therefore\ I_2 = \int \dfrac{dm}{4 + m^2 - 4m} = \int \dfrac{dm}{(m-2)^2}$

$= \dfrac{-1}{m-2} + C_2 = \dfrac{-1}{\sin\theta - 2} + C_2$

From (i), (ii) and (iii), we get

$I = 3\log(\sin\theta - 2) - \frac{6}{\sin\theta-2} + \frac{2}{\sin\theta-2} + C,$

where $C = 3C_1 - 2C_2$

$\Rightarrow\ I = 3\log(\sin\theta - 2) - \dfrac{4}{\sin\theta - 2} + C$

18. Evaluate: $\int \frac{\sin(x-a)}{\sin(x+a)} dx$

[CBSE 2015, 2013]

Solution

Let $I = \int \frac{\sin(x-a)}{\sin(x+a)} dx = \int \frac{\sin(x+a-2a)}{\sin(x+a)} dx$

$= \int \left(\dfrac{\sin(x+a)\cos 2a - \cos(x+a)\sin 2a}{\sin(x+a)}\right) dx$

$\Rightarrow I = \cos 2a \int dx - \sin 2a \int \dfrac{\cos(x+a)}{\sin(x+a)} dx$

Put $\sin(x+a) = t \Rightarrow \cos(x+a)dx = dt$

$\Rightarrow I = \cos 2a \int dx - \sin 2a \int \dfrac{dt}{t}$

$\Rightarrow\ I = x\cos 2a - \sin 2a \log|t| + C$

$\Rightarrow I = x\cos 2a - \sin 2a \log|\sin(x+a)| + C$

19. Find $\int \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx.$

[CBSE 2015, 2014, 2013C]

Solution

Let $I = \int (\sqrt{\tan x} + \sqrt{\cot x})dx$

$= \int \left(\sqrt{\frac{\sin x}{\cos x}} + \sqrt{\frac{\cos x}{\sin x}} \right) dx = \int \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}} dx$

$= \sqrt{2} \int \frac{\sin x + \cos x}{\sqrt{2 \sin x \cos x}} dx = \sqrt{2} \int \frac{\sin x + \cos x}{\sqrt{\sin 2x + 1 - 1}} dx$

$= \sqrt{2} \int \frac{\sin x + \cos x}{\sqrt{1 - (1 - \sin 2x)}} dx$

$= \sqrt{2} \int \frac{\sin x + \cos x}{\sqrt{1 - (\sin^2 x - \cos^2 x - 2 \sin x \cos x)}} dx$

$= \sqrt{2} \int \frac{\sin x + \cos x}{\sqrt{1 - (\sin x - \cos x)^2}} dx$

Put $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)dx = dt$

$\therefore\ I = \sqrt{2} \int \frac{dt}{\sqrt{1 - t^2}} = \sqrt{2} \sin^{-1} t + C$

$= \sqrt{2} \sin^{-1} (\sin x - \cos x) + C$

20. Evaluate: $\int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$

[CBSE 2014C]

Solution

Let $I = \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$

$= \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{1 - (1 - \sin 2x)}} dx$

$= \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{1 - (\sin x - \cos x)^2}} dx$

Put $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)dx = dt$

When $x = \frac{\pi}{3}$, then $t = \frac{\sqrt{3} - 1}{2} = \alpha$

and when, $x = \frac{\pi}{6}$, then $t = \frac{1 - \sqrt{3}}{2} = -\alpha$

$\therefore\ I = \int_{-\alpha}^{\alpha} \frac{dt}{\sqrt{1 - t^2}}$

$$= [\sin^{-1} t]_{-\alpha}^{\alpha} = 2\sin^{-1} \alpha = 2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

21. Evaluate: $\int_0^{\pi/4} \frac{\sin 2\theta}{\sin^4\theta+\cos^4\theta}\, d\theta$

[CBSE 2013C]

Solution

Let $I = \int_0^{\pi/4} \frac{\sin 2\theta}{\sin^4\theta+\cos^4\theta}\, d\theta$

$$= \int_0^{\pi/4} \frac{2\sin\theta\cos\theta}{\sin^4\theta + (1-\sin^2\theta)^2}\, d\theta$$

Put $\sin^2\theta = t \Rightarrow 2\sin\theta\cos\theta\, d\theta = dt$

When $\theta = 0$, then $t = 0$ and when $\theta = \frac{\pi}{4}$, then $t = \frac{1}{2}$

$$\therefore\ I = \int_0^{1/2} \frac{dt}{t^2 + (1-t)^2} = \int_0^{1/2} \frac{dt}{2t^2 - 2t + 1}$$

$$= \frac{1}{2}\int_0^{1/2} \frac{dt}{\left(t^2 - t + \frac{1}{2}\right)}$$

$$= \frac{1}{2}\int_0^{1/2} \frac{dt}{\left(t-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}\cdot\frac{1}{\frac{1}{2}}\left[\tan^{-1}\left(\frac{t-\frac{1}{2}}{\frac{1}{2}}\right)\right]_0^{1/2}$$

$$= \tan^{-1}\left[\frac{2t-1}{1}\right]_0^{1/2}$$

$$= \tan^{-1} 0 - \tan^{-1}(-1) = \frac{\pi}{4}$$

22. Evaluate: $\int_0^{\pi/4} \frac{\sin x+\cos x}{16+9\sin 2x}\, dx$

[CBSE 2018]

Solution

Let $I = \int_0^{\pi/4} \frac{\sin x+\cos x}{16+9\sin 2x}\, dx$

Put $\sin x - \cos x = t$

$$\Rightarrow\ (\cos x + \sin x)dx = dt$$

and $1 - 2\sin x\cos x = t^2 \Rightarrow 1 - \sin 2x = t^2$

When $x = \frac{\pi}{4}, t = \sin\frac{\pi}{4} - \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$

When $x = 0, t = \sin 0 - \cos 0 = -1$

$$\therefore \int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9\sin 2x} dx = \int_{-1}^{0} \frac{dt}{16 + 9(1 - t^2)}$$

$$= \int_{-1}^{0} \frac{dt}{25 - 9t^2} = \frac{1}{9} \int_{-1}^{0} \frac{dt}{\left(\frac{5}{3}\right)^2 - t^2}$$

$$= \frac{1}{9} \cdot \frac{1}{2 \times \frac{5}{3}} \left[\log \left[\frac{\frac{5}{3} + t}{\frac{5}{3} - t} \right| \right]_{-1}^{0}$$

$$= \frac{1}{30} [\log 1 - (\log 1 - \log 4)] = \frac{1}{30} \log 4$$

23. Evaluate: $\int_1^3 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{4-x}} dx$

[CBSE 2021-22]

Solution

$$\int_1^3 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{4 - x}} dx$$

Let $I = \int_1^3 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{4-x}} dx$

$I = \int_1^3 \frac{\sqrt{4-x}}{\sqrt{4-x}+\sqrt{x}} dx$ ... (ii) $\left[\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$

Adding (i) and (ii), we get

$$2l = \int_1^3 \frac{\sqrt{x} + \sqrt{4 - x}}{\sqrt{x} + \sqrt{4 - x}} dx$$

$\Rightarrow 2l = \int_1^3 1dx \Rightarrow 2l = [x]_1^3$

$\Rightarrow 2I = 2 \Rightarrow I = 1$

24. Evaluate: $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$

[CBSE 2017, 2014, 2014C]

Solution

Let $I = \int_0^{\pi} \frac{x\tan x}{\sec x+\tan x} dx$

$$= \int_0^{\pi} \frac{x \frac{\sin x}{\cos x}}{\frac{\sin x}{\cos x} + \frac{\cos x}{\cos}} dx = \int_0^{\pi} \frac{x\sin x}{1 + \sin x} dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \sin(\pi - x)} dx$$

$$\left[\text{Using the property } \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \sin x} dx$$

Adding (i) and (ii), we get

$$
\begin{aligned}
2I &= \int_0^{\pi} \frac{\pi \sin x}{1+\sin x}\,dx \\
I &= \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\sin x}\,dx = \frac{\pi}{2}\int_0^{\pi} \frac{1+\sin x-1}{1+\sin x}\,dx \\
&= \frac{\pi}{2}\int_0^{\pi} \left(1-\frac{1}{1+\sin x}\right)dx \\
&= \frac{\pi}{2}\left[\int_0^{\pi} dx - \int_0^{\pi} \frac{1}{1+\sin x}\,dx\right] \\
&= \frac{\pi}{2}\left[[x]_0^{\pi} - \int_0^{\pi} \frac{1-\sin x}{\cos^2 x}\,dx\right] \\
&= \frac{\pi}{2}\left[[x]_0^{\pi} - \int_0^{\pi} \left(\frac{1}{\cos^2 x}-\frac{\sin x}{\cos^2 x}\right)dx\right] \\
&= \frac{\pi}{2}\left[(\pi-0) - \int_0^{\pi} (\sec^2 x - \tan x \cdot \sec x)dx\right] \\
&= \frac{\pi}{2}[\pi - (\tan x - \sec x)_0^{\pi}] \\
&= \frac{\pi}{2}[\pi - [(0-(-1)) - (0-1)]] = \frac{\pi}{2}[\pi - 2]
\end{aligned}
$$

25. Evaluate: $\int_0^{\pi/2} \frac{dx}{1+\sqrt{\tan x}}$

[CBSE 2015C]

Solution

Let $I = \int_0^{\pi/2} \frac{dx}{1+\sqrt{\tan x}} = \int_0^{\pi/2} \frac{dx}{1+\sqrt{\frac{\sin x}{\cos x}}}$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

By the property, $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, we get

$$
\begin{aligned}
I &= \int_0^{\pi/2} \frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}\,dx \\
&= \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx
\end{aligned}
$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \left[\frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} + \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\right]dx$$

$$= \int_0^{\pi/2} 1 \cdot dx = [x]_0^{\pi/2} = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

26. Evaluate: $\int_0^{\pi/4} \log(1+\tan x)\, dx$

[CBSE 2015C]

Solution

Let $I = \int_0^{\pi/4} \log(1+\tan x) dx$

By using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, we get

$I = \int_0^{\pi/4} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right] dx$

$= \int_0^{\pi/4} \log\left[1+\frac{1-\tan x}{1+\tan x}\right] dx = \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right) dx$

$= \int_0^{\pi/4} [\log 2 - \log(1+\tan x)] dx$

$= \log 2 \int_0^{\pi/4} 1 \cdot dx - \int_0^{\pi/4} \log(1+\tan x) dx = \log 2 [x]_0^{\pi/4} - 1$

$\Rightarrow 2I = \log 2 \left[\frac{\pi}{4} - 0\right] \Rightarrow I = \frac{\pi}{8} \log 2$

27. Evaluate: $\int_{-1}^{2} |x^3 - 3x^2 + 2x|\, dx$

[CBSE SQP 2021-22]

Solution

The given definite integral

$$= \int_{-1}^{2} |x(x-1)(x-2)| dx$$

$$= \int_{-1}^{0} |x(x-1)(x-2)| dx + \int_{0}^{1} |x(x-1)(x-2)| dx + \int_{1}^{2} |x(x-1)(x-2)| dx$$

$$= -\left[\frac{x^4}{4} - x^3 + x^2\right]_{-1}^{0} + \left[\frac{x^4}{4} - x^3 + x^2\right]_{0}^{1} - \left[\frac{x^4}{4} - x^3 + x^2\right]_{1}^{2}$$

$$= \frac{9}{4} + \frac{1}{4} + \frac{1}{4} = \frac{11}{4}$$

28. Evaluate: $\int_0^{3/2} |x \sin \pi x|\, dx$

[CBSE 2017]

Solution

Let $I = \int_0^{3/2} |x\sin\pi x| dx$

When $0 < x < 1 \Rightarrow 0 < \pi x < \pi \Rightarrow \sin\pi x > 0$

When $1 < x < \frac{3}{2} \Rightarrow \pi < \pi x < \frac{3\pi}{2} \Rightarrow \sin\pi x < 0$

$$\therefore |x\sin\pi x| = \begin{cases} x\sin\pi x, \text{ if } 0 < x < 1 \\ -x\sin\pi x, \text{ if } 1 < x < \frac{3}{2} \end{cases}$$

$$\therefore I = \int_0^1 x\sin\pi x dx - \int_1^{3/2} x\sin\pi x dx$$

$$= \left[\frac{-x\cos\pi x}{\pi}\right]_0^1 + \int_0^1 \frac{\cos\pi x}{\pi} dx + \left[\frac{x\cos\pi x}{\pi}\right]_1^{3/2} - \int_1^{3/2} \frac{\cos\pi x}{\pi} dx$$

$$= \frac{1}{\pi} - 0 + \left[\frac{\sin\pi x}{\pi^2}\right]_0^1 + 0 - \frac{(-1)}{\pi} - \left[\frac{\sin\pi x}{\pi^2}\right]_1^{3/2}$$

$$= \frac{2}{\pi} + 0 - \frac{(-1)}{\pi^2} + 0 = \frac{2}{\pi} + \frac{1}{\pi^2}$$

29. If $\frac{d}{dx}(f(x)) = \log x$, then $f(x)$ equals:

(a) $-\frac{1}{x} + C$

(b) $x(\log x - 1) + C$

(c) $x(\log x + x) + C$

(d) $\frac{1}{x} + C$

[CBSE 2023]

Solution

(b): We have, $\frac{d}{dx}(f(x)) = \log x$

On integrating both sides, we get $\int \frac{d}{dx}(f(x)) = \int 1 \cdot \log x dx$

$\Rightarrow f(x) = \log x \int 1 \cdot dx - \int \left(\frac{d}{dx}(\log x) \int 1 \cdot dx\right) dx$

[Integrating by parts]

$\Rightarrow f(x) = x \cdot \log x - \int \frac{1}{x} x dx \Rightarrow f(x) = x\log x - x + C$

$\Rightarrow \mathrm{f(x)} = \mathrm{x(\log x - 1)} + C$

30. $\int e^{5\log x} dx$ is equal to

(a) $\frac{x^5}{5} + C$

(b) $\frac{x^6}{6} + C$

(c) $5x^4 + C$

(d) $6x^5 + C$

[CBSE 2023]

Solution

. (b):Let $I = \int e^{5\log x} dx = \int e^{\log x^5} dx = \int x^5 dx \left[\because e^{\log x} = x\right]$

$$= \frac{x^6}{6} + C$$

31. Find: $\int \frac{2x^2+1}{x^2(x^2+4)} dx$

[CBSE 2023]

Solution

Let $I = \int \frac{2x^2+1}{x^2(x^2+4)} dx$

Using partial fraction, $\frac{2x^2+1}{x^2(x^2+4)} = \frac{A}{x^2} + \frac{B}{x^2+4}$

$\Rightarrow 2x^2 + 1 = A(x^2 + 4) + Bx^2$

Comparing the coefficient of x^2 and x^0, we get

$$(\ldots\ldots)A + B = 2$$

and $4A = 1 \Rightarrow A = 1/4$

From (i), $B = 7/4$

So, $I = \int \frac{2x^2+1}{x^2(x^2+4)} dx = \int \frac{1}{4x^2} dx + \int \frac{7}{4(x^2+4)} dx$

$= \frac{1}{4}\int \frac{1}{x^2} dx + \frac{7}{4}\int \frac{dx}{x^2+2^2} = \frac{-1}{4x} + \frac{7}{4} \times \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$

$$= \frac{-1}{4x} + \frac{7}{8}\tan^{-1}\left(\frac{x}{2}\right) + C$$

32. $\int_{-1}^{1} \frac{|x-2|}{x-2} dx, x \neq 2$ is equal to

(a) 1

(b) -1

(c) 2

(d) -2

[CBSE 2023]

Solution

(d): Let $I = \int_{-1}^{1} \frac{|x-2|}{x-2} dx$

$= \int_{-1}^{1} \frac{-(x-2)}{x-2} dx = \int_{-1}^{1} -1 \cdot dx = [-x]_{-1}^{1} = -[1 - (-1)] = -2$

33. Evaluate: $\int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x + \cos^4 x} dx$

[CBSE 2023]

Solution

$$\text{Let } I = \int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x + \cos^4 x} dx$$

$$\therefore I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)\sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)}{\sin^4\left(\frac{\pi}{2} - x\right) + \cos^4\left(\frac{\pi}{2} - x\right)} dx \#(i)$$

$$= \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right)\sin x\cos x}{\cos^4 x + \sin^4 x} dx \#(ii)$$

Adding (i) and (ii), we get

$$21 = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x\cos x}{\sin^4 x + \cos^4 x} dx$$

Dividing numerator and denominator by $\cos^4 \boldsymbol{x}$, we get

$$21 = \frac{\pi}{2}\int_0^{\pi/2} \frac{\tan x \sec^2 x}{1+\tan^4 x}\,dx$$

Put $\tan^2 x = t \Rightarrow 2\tan x \sec^2 x dx = dt$

When $x = 0$, then $t = 0$ and when $x = \frac{\pi}{2}$, then $t = \infty$

$\therefore 1 = \frac{\pi}{8}\int_0^{\infty} \frac{dt}{1+t^2} = \frac{\pi}{8}|\tan^{-1} t|_0^{\infty} = \frac{\pi}{8} \times \frac{\pi}{2} = \frac{\pi^2}{16}$

34. Evaluate : $\int_0^{2\pi} \frac{1}{1+e^{\sin x}}\,dx$

[CBSE 2023]

Solution

I. $\int_0^{2\pi} \frac{1}{1+e^{2\pi \cdot 1}}\,dx - (1) =$

$r \cdot \int_0^{\pi} \frac{1}{1+e^{\sin(2\pi),dx}}$

$\{\because \sqrt[a]{f(x)}dx \cdot \sqrt[a]{f(a \cdot x)da}\}$

I. $\int_0^{\pi} \frac{1}{1+e^{\sin x}}\,dx \cdot \int_0^{2\pi} \frac{e^{\sin x}}{L+e^{\sin x}}\,dx$-(1)

$2I \cdot \int_0^{2\pi} \frac{1+e^{\cos x}}{1+e^{\sin x}}\,dx = \int_0^{2\pi} 1dx$

$2I = [x]_0^{2\pi}$

$2I = 2r$

I $\cdot \pi$

ans π

35. Find: $\int \frac{e^{4x}-1}{e^{4x}+1}\,dx$

[CBSE 2024]

Solution

if $I = \int \frac{(e^{6x}-1)}{e^{6x}+1}\,dx$

Divide Numeraton and Donominater by buy 6^{22}

so.

$I = \int \frac{(e^{2x}-e^{-2x})dx}{e^{2x}.e^{-2x}}$

Take

$dv = (\alpha e^{2x} = 2e^{-2x})dx$

So, $\frac{du}{2} = (e^{2x} - e^{-2x})dx$

Hana. $I = \frac{1}{2}\int \frac{d\sigma}{v} = \frac{1}{2}\log_e |v| + c$

Ans: $I = \frac{1}{2}\log_e|e^{2x} + e^{-2x}| + c$

[Topper's Answer, 2024]

36. Find: $\int e^x \left[\frac{1}{(1+x^2)^{\frac{3}{2}}} + \frac{x}{\sqrt{1+x^2}}\right] dx$

[CBSE 2024]

Solution

Let $I = \int e^x \left[\frac{1}{(1+x^2)^{3/2}} + \frac{x}{\sqrt{1+x^2}}\right] dx$

Let $f(x) = \frac{x}{\sqrt{1+x^2}}$ and $f'(x) = \frac{1}{(1+x^2)^{3/2}}$

We know that $\int e^x[f(x) + f'(x)]dx = f(x)e^x + C$

$$\therefore I = e^x \frac{x}{\sqrt{1+x^2}} + C\#(i)$$

37. Evaluate: $\int_{-2}^{2} \sqrt{\frac{2-x}{2+x}}\,dx$

[CBSE 2024]

Solution

I. $\int_{-2}^{2} \sqrt{\frac{2-x}{2+x}}\,dx$

Tale $x = 2\,\text{coso}$

$\text{dx} = -2\,\text{sinade}$

The limits change

x	2	-2
θ	0	π

I. $\int_{\pi}^{0} \sqrt{\frac{2-2\cos\theta}{2+2\cos\theta}}(-2\sin\theta d\theta)$

$I = \int_{0}^{\pi} \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} - (-2\sin\theta d\theta) I = \int_{\pi}^{\pi} \sqrt{\frac{1\sin^2\theta/2}{1\cos^2\theta/2}} \cdot (-2\sin\theta d\theta)$

I. $\int_{\pi}^{0} \tan\theta/2(-2\sin\theta d\theta)$

Using property $\int_a^b f(x)dx \cdot -\int_b^a f(x)dx$

$I = \int_0^{\pi} (\tan\theta/2)(2\sin\theta d\theta)[\sin\theta - 2\sin\theta/2\cos\theta/2] I = \int_0^{\pi} \frac{\sin\theta/2}{\cos\theta/2}(+\theta\sin\theta/2\cos\theta/2 d\theta) I$

$= +\int_0^{\pi} \sin^2 0/2 d0$

I. $4\int_0 \left(\frac{1-\cos\theta}{2}\right) d\theta$

$I = 2\int_0^{\pi} (1-\cos\theta) d\theta$

- $2(\theta - \sin\theta)1_0^{\pi}$
  $= 2((\pi - \sin\pi))^{-10-\sin 01}) = 2\pi$
  [Topper's Answer, 2024]

38. Evaluate: $\int_0^{\frac{\pi}{4}} \frac{xdx}{1+\cos 2x+\sin 2x}$

[CBSE 2024]

Solution

Let $I = \int_0^{\pi/4} \frac{xdx}{1+\cos 2x+\sin 2x}$

$$\Rightarrow I = \int_0^{\pi/4} \frac{\left(\frac{\pi}{4}-x\right)}{1+\cos 2\left(\frac{\pi}{4}-x\right)+\sin 2\left(\frac{\pi}{4}-x\right)} dx\#(i)$$

$$\Rightarrow I = \int_0^{\pi/4} \frac{\left(\frac{\pi}{4}-x\right)}{1+\sin 2x+\cos 2x} dx\#(ii)$$

Adding (i) and (ii), we get

$$21 = \frac{\pi}{4}\int_0^{\pi/4} \frac{dx}{1+\sin 2x+\cos 2x}$$

$$\Rightarrow 21 = \frac{\pi}{4}\int_0^{\pi/4} \frac{1+\tan^2 x}{1+\tan^2 x+2\tan x+1-\tan^2 x}dx$$

$$\Rightarrow I = \frac{\pi}{8}\int_0^{\pi/4} \frac{\sec^2 x}{2(1+\tan x)}dx$$

Put $1+\tan x = t \Rightarrow \sec^2 x dx = dt$

$l = \frac{\pi}{8}\int_1^2 \frac{dt}{2t} = \frac{\pi}{16}[\log t]_1^2 = \frac{\pi}{16}\log 2$

39. Find: $\int \frac{1}{x}\sqrt{\frac{x+a}{x-a}}dx$.

[CBSE 2025]

Solution

. Let $I = \int \frac{1}{x}\sqrt{\frac{x+a}{x-a}}dx$

$$= \int \frac{1}{x}\sqrt{\frac{(x+a)(x+a)}{(x-a)(x+a)}}dx = \int \frac{(x+a)}{x\sqrt{x^2-a^2}}dx$$

$$= \int \frac{1}{\sqrt{x^2-a^2}}dx + a\int \frac{1}{x\sqrt{x^2-a^2}}dx$$

Here, $I_1 = a\int \frac{1}{x\sqrt{x^2-a^2}}dx = a\int \frac{x}{x^2\sqrt{x^2-a^2}}dx$

Let $x^2-a^2 = t^2 \Rightarrow xdx = tdt$

$$I_1 = a\int \frac{t}{(t^2+a^2)\cdot t}dt = a\int \frac{1}{(t^2+a^2)}dt$$

$$= a\left(\frac{1}{a}\tan^{-1}\left(\frac{t}{a}\right)\right) = \tan^{-1}\left(\frac{\sqrt{x^2-a^2}}{a}\right)$$

Substitute the value of l_1 in (i), we get

$$\therefore I = \log\left|x+\sqrt{x^2-a^2}\right| + \tan^{-1}\left(\frac{\sqrt{x^2-a^2}}{a}\right) + c$$

40. Find: $\int \sin^{-1}\sqrt{\frac{x}{a+x}}dx$.

[CBSE 2025]

Solution

Let $I = \int \sin^{-1}\sqrt{\frac{x}{a+x}}dx$

Let $x = a\tan^2\theta \Rightarrow dx = 2a\tan\theta\sec^2\theta\cdot d\theta$

$\therefore I = \int \sin^{-1}\sqrt{\frac{a\tan^2\theta}{a+a\tan^2\theta}}(2a\tan\theta\sec^2\theta)d\theta$

$$\therefore I = \int \sin^{-1}\sqrt{\frac{\tan^2\theta}{1+\tan^2\theta}}(2a\tan\theta\sec^2\theta)d\theta$$

$$I = \int \sin^{-1}(\sin\theta)(2a\tan\theta\sec^2\theta)d\theta[\because 1+\tan^2\theta = \sec^2\theta]$$

$$\Rightarrow I = 2a\int \theta\cdot\tan\theta\sec^2\theta\cdot d\theta \#(i)$$

Let $u = \theta$ and $dv = \tan\theta\sec^2\theta\cdot d\theta$

$\therefore du = d\theta$ and $v = \frac{\tan^2\theta}{2}$ $\therefore I = 2a\int u\cdot dv$ [From (i)]

$= 2a[u\cdot v - \int v\cdot du]$;

$I = 2a\left[\theta\cdot\frac{\tan^2\theta}{2} - \int\frac{\tan^2\theta}{2}d\theta\right]$

$$I = 2a\left[\frac{\theta\cdot\tan^2\theta}{2} - \frac{1}{2}\int(\sec^2\theta - 1)d\theta\right]$$

$$= 2a\left[\frac{\theta\tan^2\theta}{2} - \frac{1}{2}(\tan\theta - \theta)\right] + C = a[\theta\tan^2\theta - \tan\theta + \theta] + C$$

$$= a\left[\frac{x}{a}\cdot\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{\frac{x}{a}} + \tan^{-1}\sqrt{\frac{x}{a}}\right] + C$$

$$= (x+a)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + C$$

Now, $I_1 = \int\frac{\sin\theta\cos\theta}{4+\sin^2\theta-4\sin\theta}d\theta$

41. Evaluate: $\int_0^{\pi/2}\frac{x}{\sin x+\cos x}dx$

[CBSE 2025]

Solution

. Let $I = \int_0^{\pi/2}\frac{x}{\sin x+\cos x}dx$

$$= \int_0^{\pi/2}\frac{\frac{\pi}{2}-x}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}dx\cdot\left[\because\int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]\#(i)$$

$$\Rightarrow I = \int_0^{\pi/2}\frac{\frac{\pi}{2}-x}{\cos x+\sin x}dx\#(ii)$$

Adding equation (i) and (ii), we get

$2l = \int_0^{\pi/2}\frac{\frac{\pi}{2}}{\sin x+\cos x}dx \Rightarrow l = \frac{\pi}{4}\int_0^{\pi/2}\frac{1}{\sin x+\cos x}dx$

$= \frac{\pi}{4}\int_0^{\pi/2}\frac{1}{\frac{2\tan(x/2)}{1+\tan^2(x/2)}+\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}}dx$

$$= \frac{\pi}{4}\int_0^{\pi/2}\frac{1+\tan^2(x/2)}{2\tan(x/2)+1-\tan^2(x/2)}dx$$

$= \frac{\pi}{4}\int_0^{\pi/2}\frac{\sec^2(x/2)}{2\tan(x/2)+1-\tan^2(x/2)}dx$

Put $\tan\left(\frac{x}{2}\right) = t \Rightarrow \frac{\sec^2(x/2)}{2}dx = dt$

When $x = 0$, then $t = 0$ and when $x = \frac{\pi}{2}$, then $t = 1$

$$\therefore I = \frac{\pi}{4}\int_0^1 \frac{2dt}{2t+1-t^2} = \frac{\pi}{2}\int_0^1 \frac{dt}{(\sqrt{2})^2-(t-1)^2}$$

$$= \frac{\pi}{2} \times \frac{1}{2\sqrt{2}}\left[\log\left|\frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}\right|\right]_0^1 = \frac{\pi}{4\sqrt{2}}\left\{\log\left(\frac{\sqrt{2}}{\sqrt{2}}\right) - \log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right\}$$

$$= \frac{\pi}{4\sqrt{2}}\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = \frac{\pi}{4\sqrt{2}}\log(\sqrt{2}+1)^2$$

$$I = \frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)$$

Application of Integrals

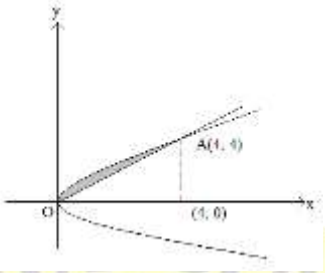
1. Using integration, find the area of the region enclosed by the curves $y^2 = 4x$ and $y = x$.

[CBSE 2013C]

Solution

The given curves are $y^2 = 4x, y = x$

They intersect at $O(0,0)$ and $A(4,4)$.



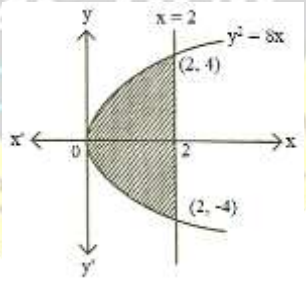
$\therefore$ Required area $= \int_0^4 (\sqrt{4x} - x)dx = \left[2 \cdot \frac{x^{3/2}}{3/2} - \frac{x^2}{2}\right]_0^4$

...(i) $= \frac{4}{3} \cdot 4^{3/2} - \frac{1}{2} \cdot 4^2 = \frac{32}{3} - 8 = \frac{8}{3}$ sq.units.

2. Find the area of the region bounded by the parabola $y^2 = 8x$ and the line $x = 2$.

[CBSE SQP 2020-21]

Solution



Required area

$$= 2\int_0^2 \sqrt{8x}dx$$
$$= 2 \times 2\sqrt{2}\int_0^2 x^{1/2}dx$$
$$= 4\sqrt{2}\left[\frac{2}{3}x^{3/2}\right]_0^2$$

$= \frac{8}{3}\sqrt{2}\left[2^{3/2} - 0\right] = \frac{8\sqrt{2}}{3} \times 2\sqrt{2} = \frac{32}{3}$ sq. units

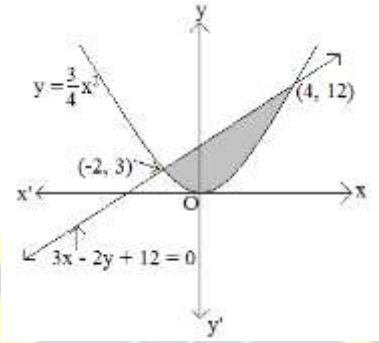
3. Find the area enclosed between the parabola $4y = 3x^2$ and the straight line $3x - 2y + 12 = 0$.

[CBSE 2017,2015C]

Solution

Given equations are $y = \frac{3x^2}{4}$

and $3x - 2y + 12 = 0 \Rightarrow y = \frac{3x+12}{2}$



Solving equations (i) and (ii), we get

$$\frac{3x^2}{4} = \frac{3x + 12}{2}$$

$$\Rightarrow x^2 - 2x - 8 = 0 \Rightarrow (x + 2)(x - 4) = 0$$

$$\Rightarrow x = -2, 4$$

When $x = -2 \Rightarrow y = 3$

When $x = 4 \Rightarrow y = 12$

$$\therefore \text{ Required area } = \int_{-2}^{4} \left(\frac{3x + 12}{2} - \frac{3}{4}x^2\right) dx$$

$$= \left[\frac{3}{4}x^2 + 6x - \frac{x^3}{4}\right]_{-2}^{4}$$

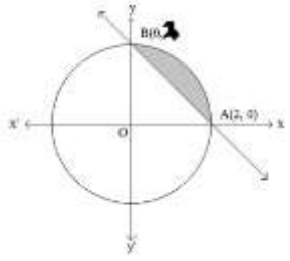
$$= \left[\frac{3 \times 16}{4} + 6 \times 4 - \frac{64}{4}\right] - \left[\frac{3}{4} \times 4 - 6 \times 2 + \frac{8}{4}\right]$$

$= 27$ sq. units.

4. Using integration, find the smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$.

[CBSE 2020]

Solution



The given curves are $x^2 + y^2 = 4$ ... (i) and $x + y = 2$

$\therefore$ Required area $= \int_0^2 \left[\sqrt{4 - x^2} - (2 - x)\right] dx$

$$= \left[\frac{x\sqrt{4 - x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2} - 2x + \frac{x^2}{2}\right]_0^2$$

$$= 0 + 2\sin^{-1}(1) - 4 + 2 - 0$$

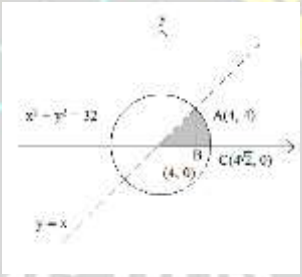
$$= 2\frac{\pi}{2} - 2 = (\pi - 2) \text{ sq. units.}$$

$$= \int \sqrt{a^2 - x^2}\,dx = \frac{x \cdot \sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$$

5. Using integration, find the area of the region in the first quadrant enclosed by the x-axis, the line y = x and the circle $x^2 + y^2 = 32$.

[CBSE 2018, 2014]

Solution



. The given equation of the circle is $x^2 + y^2 = 32$ and the line is $y = x$

These intersect at $A(4,4)$ in the first quadrant. The required area is shown shaded in the figure.

Points $B(4,0)$ and $C(4\sqrt{2}, 0)$

$\therefore$ Required area = Area BACB + Area OABO

$$= \int_4^{4\sqrt{2}} y_1\,dx + \int_0^4 y_2\,dx = \int_4^{4\sqrt{2}} \sqrt{32 - x^2}\,dx + \int_0^4 x\,dx$$

$$= \int_4^{4\sqrt{2}} \sqrt{(4\sqrt{2})^2 - x^2}\,dx + \int_0^4 x\,dx$$

$$= \left[\frac{x\sqrt{32 - x^2}}{2} + \frac{32}{2}\sin^{-1}\left(\frac{x}{4\sqrt{2}}\right)\right]_4^{4\sqrt{2}} + \left[\frac{x^2}{2}\right]_0^4$$

$$= \frac{4\sqrt{2} \times 0}{2} + 16\sin^{-1} 1 - \left(\frac{4 \times 4}{2} + 16\sin^{-1}\frac{1}{\sqrt{2}}\right) + \frac{1}{2}(4^2 - 0)$$

(ii) $= 16 \cdot \frac{\pi}{2} - \left(8 + 16 \cdot \frac{\pi}{4}\right) + 8 = 16\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = 4\pi$ sq.units

6. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the line $\frac{x}{3} + \frac{y}{2} = 1$.

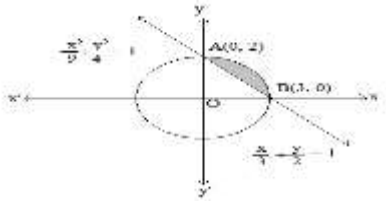
[CBSE 2014]

Solution

We have $\frac{x^2}{9}+\frac{y^2}{4}=1$ ... (i) and $\frac{x}{3}+\frac{y}{2}=1$

Curve (i) is an ellipse of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

That means its major axis is along x - axis. Also this ellipse is symmetrical about the x - axis.



$$\text{Required area} = \frac{2}{3}\int_0^3 \sqrt{(3)^2-x^2}\,dx - \frac{2}{3}\int_0^3 (3-x)\,dx$$

$$= \frac{2}{3}\left[\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3 - \frac{2}{3}\left[\frac{(3-x)^2}{-2}\right]_0^3$$

$$= \frac{2}{3}\left[\left(0+\frac{9}{2}\sin^{-1}(1)\right)-\left(0+\frac{9}{2}\sin^{-1}(0)\right)\right]+\frac{1}{3}[0^2-9]$$

$$= \frac{3\pi}{2}-3 \text{ sq. units.}$$

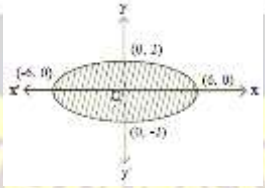
7. Find the area of the ellipse $x^2+9y^2 = 36$ using integration.

[CBSE SQP 2020-21]

Solution

Given equation of ellipse is

$$x^2+9y^2=36$$



∴ Required area

$$= 4\int_0^6 \sqrt{\frac{36-x^2}{9}}\,dx$$

$$= \frac{4}{3}\int_0^6 \sqrt{6^2-x^2}\,dx$$

$$= \frac{4}{3}\left[\frac{x}{2}\sqrt{6^2-x^2}+18\sin^{-1}\left(\frac{x}{6}\right)\right]_0^6$$

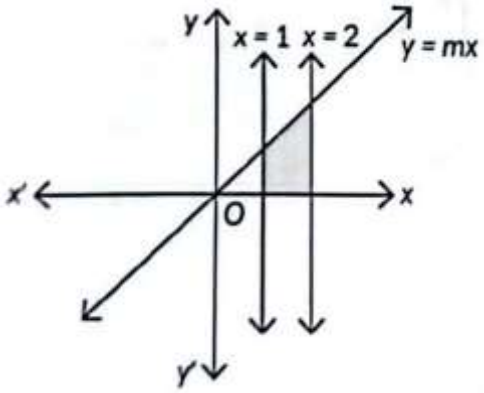
$$= \frac{4}{3}\left[18\times\frac{\pi}{2}-0\right] = 12\pi \text{ sq. units}$$

$$= \frac{4}{3}\left[18\times\frac{\pi}{2}-0\right] = 12\pi \text{ sq. units}$$

8. Using integration, find the area of the region bounded by $y = mx (m > 0), x = 1, x = 2$ and the x-axis.

[CBSE 2023]

Solution



We have, $y = mx$

From the figure it is clear that required area is the shaded region.

$$\text{Required area } = \int_1^2 mx\,dx = \left[\frac{mx^2}{2}\right]_1^2$$

$$= \frac{4m}{2} - \frac{m}{2} = \frac{3m}{2} \text{ sq. units.}$$

9. The area (in sq. units) of the region bounded by the curve $y = x, x$-axis, $x = 0$ and $x = 2$ is :

(a) $\frac{3}{2}$

(b) $\frac{1}{2}\log 2$

(c) 2

(d) 4

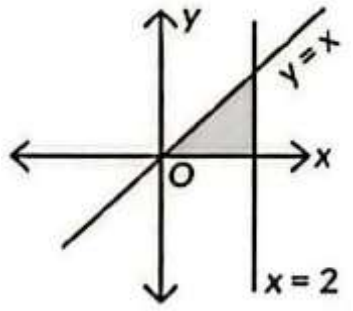
[CBSE 2024]

Solution

(c): Given, $y = x$

$$\text{Required Area } = \int_0^2 x\,dx = \left[\frac{x^2}{2}\right]_0^2$$

$$= \left(\frac{4}{2} - 0\right) = 2 \text{ sq. units}$$



10. The area bounded by the curve $y = \sqrt{x}, Y$-axis and between the lines $y = 0$ and $y = 3$ is

(a) $2\sqrt{3}$

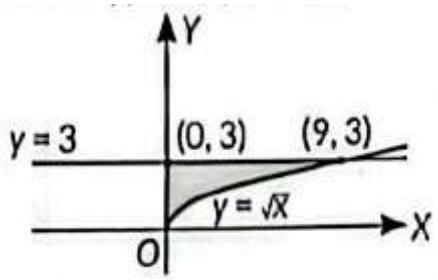
(b) 27

(c) 9
(d) 3

[CBSE 2024]

Solution

(c) : Given, $y = \sqrt{x}$
and $y = 3$
Intersection points of (i) and (ii) is (9,3).



Required area; $A = \int_0^3 y^2 dy$

$$\Rightarrow A = \left[\frac{y^3}{3}\right]_0^3 = \frac{27}{3} - 0 = 9 \text{ sq. units}$$

11. Find the area of the region bounded by the curve $4x^2 + y^2 = 36$ using integration.

[CBSE 2024]

Solution

Given, equation of curve $4x^2 + y^2 = 36$

$$\Rightarrow \frac{x^2}{9} + \frac{y^2}{36} = 1$$

which is an ellipse with centre $(0,0)$.
$\therefore$ Area bounded by the ellipse

$$= 4\int_0^3 |y| dx$$

$$= 4\int_0^3 2\sqrt{9 - x^2} dx = 8\int_0^3 \sqrt{(3)^2 - x^2} dx$$

$$= 8\left[\frac{x}{2}\sqrt{9 - x^2} + \frac{3^2}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3$$

$$\left(\because \int \sqrt{a^2 - x^2} dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right)$$

$$= 8\left[0 + \frac{9}{2}\sin^{-1}\left(\frac{3}{3}\right) - 0\right] = 8 \times \frac{9}{2}\sin^{-1}(1)$$

$$= 8 \times \frac{9}{2} \times \frac{\pi}{2} = 18\pi \text{ sq. units}$$

12. Sketch the graph of $y = x|x|$ and hence find the area bounded by this curve, X-axis and the ordinates $x = -2$ and $x = 2$, using integration.

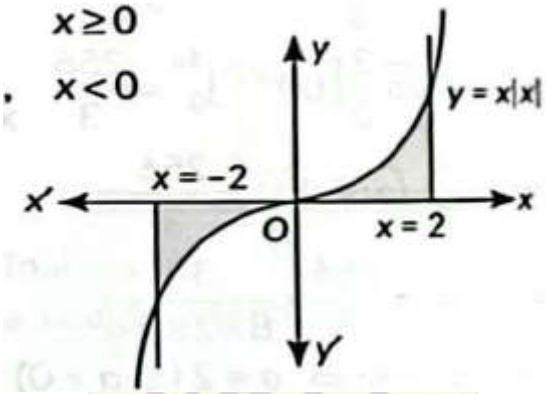
[CBSE 2024]

Solution

We have, $y = x|x| = \begin{cases} x^2, x \geq 0 \\ -x^2, x < 0 \end{cases}$

Required area $= 2\int_0^2 x^2 dx$

$= 2\left[\frac{x^3}{3}\right]_0^2 = \frac{16}{3}$ sq.units



13. Using integration, find the area bounded by the ellipse $9x^2 + 25y^2 = 225$, the lines $x = -2, x = 2$, and the X-axis.

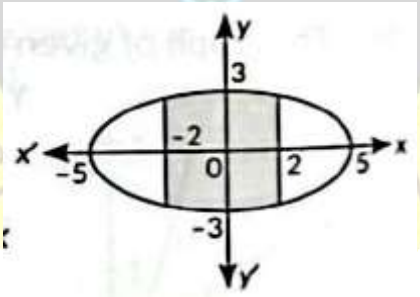
[CBSE 2024]

Solution

We have, $9x^2 + 25y^2 = 225$

$\Rightarrow \frac{x^2}{25} + \frac{y^2}{9} = 1$

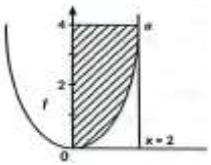
Required area $= 2 \times \frac{3}{5}\int_{-2}^{2}\sqrt{25 - x^2}dx$



$= \frac{2\times6}{5}\int_0^2\sqrt{25 - x^2}dx = \frac{12}{5}\left[\frac{x}{2}\sqrt{25 - x^2} + \frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right)\right]_0^2$

$= \frac{12}{5}\left[\sqrt{21} + \frac{25}{2}\sin^{-1}\left(\frac{2}{5}\right)\right]$ sq.units

14. The area of the shaded region (figure) represented by the curves $y = x^2, 0 \leq x \leq 2$ and y-axis is given by



(a) $\int_0^2 x^2 dx$

(b) $\int_0^2 \sqrt{y}dy$

(c) $\int_0^4 x^2 dx$

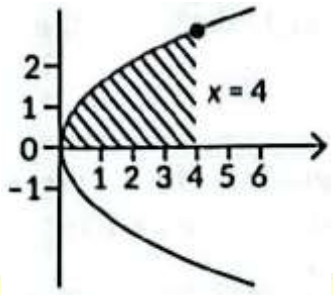
(d) $\int_0^4 \sqrt{y} dy$

[CBSE 2025]

Solution

(d)

15. The area of the shaded region bounded by the curves $y^2 = x, x = 4$ and the x-axis is given by



(a) $\int_0^4 x dx$

(b) $\int_0^2 y^2 dy$

(c) $2\int_0^4 \sqrt{x} dx$

(d) $\int_0^4 \sqrt{x} dx$

Solution

(d)

16. Use integration to find the area of the region enclosed by curve $y = -x^2$ and the straight lines $x = -3, x = 2$ and $y = 0$. Sketch a rough figure to illustrate the bounded region.

[CBSE 2025]

Solution

Given, the equation of curves

$y = x^2$ or $x^2 = -y$

$x = -3, x = 2$ and $y = 0$

$\therefore$ Area of the region bounded by the given curves

$$= \left|\int_{-3}^{0} (-x^2) dx\right| + \left|\int_{0}^{2} (-x^2) dx\right| = \left|\left[\frac{-x^3}{3}\right]_{-3}^{0}\right| + \left|\left[\frac{-x^3}{3}\right]_{0}^{2}\right|$$

$$= \frac{1}{3}[0^3 - (-3)^3] + \frac{1}{3}[2^3 - 0^3]$$

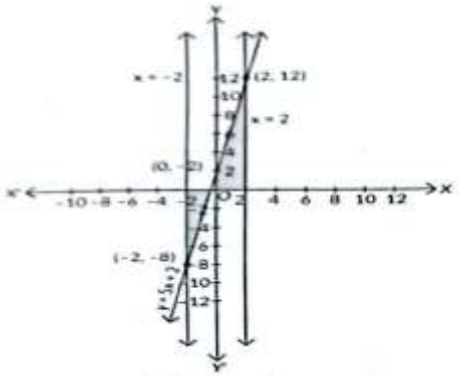
$$= \frac{27}{3} + \frac{8}{3} = \frac{35}{3} \text{ sq. units}$$

17. Using integration, find the area of the region bounded by the line $y = 5x + 2$, the $x -$ axis and the ordinates $x = -2$ and $x = 2$.

[CBSE 2025]

Solution

Given, the line $y = 5x + 2$, and the ordinates are $x = -2$ and $x = 2$.



∴ Required area $= \int_{-2}^{2} y dx = \int_{-2}^{2} (5x + 2) dx$

$$= \left[5\left(\frac{x^2}{2}\right) + 2x\right]_{-2}^{2}$$

$$= \left(5\left(\frac{(2)^2}{2}\right) + 2(2)\right) - \left(5\left(\frac{(-2)^2}{2}\right) + 2(-2)\right)$$

$= 10 + 4 - 10 + 4 = 8$ sq. units

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Differential Equations

1. Write the sum of the order and the degree of the following differential equation. $\frac{d}{dx}\left(\frac{dy}{dx}\right) = 5$

[CBSE SQP 2021-22]

Solution

We have, $\frac{d}{dx}\left(\frac{dy}{dx}\right) = 5 \Rightarrow \frac{d^2y}{dx^2} = 5$

$\therefore$ Order $= 2$

Degree $= 1$

$\therefore$ Required sum $= 3$

2. Find the order and the degree of the differential equation $x^2\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^4$.

[CBSE 2019]

Solution

The given differential equation is

$$x^2\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^4$$

$\therefore$ Its order is 2 and degree is 1 .

3. Write the degree of the differential equation: $\left(\frac{dy}{dx}\right)^4 + 3y\frac{d^2y}{dx^2} = 0$

[CBSE 2013C]

Solution

The degree of the differential equation is 1.

4. Write the general solution of differential equation $\frac{dy}{dx} = e^{x+y}$.

[CBSE SQP 2019-20]

Solution

. We have, $\frac{dy}{dx} = e^{x+y}$

$\Rightarrow \frac{dy}{dx} = e^x e^y \Rightarrow \frac{dy}{e^y} = e^x dx$

Integrating both sides, we get

$$-e^{-y} + c = e^x \Rightarrow e^x + e^{-y} = c$$

5. Find the particular solution of the differential equation $cos\ y\ dx\ +\ (1 + e^{-x})\ sin\ y\ dy\ =\ 0$ given that $y = \frac{\pi}{4}$ when x = 0.

[CBSE 2020]

Solution

We have, $\cos y dx + (1 + e^{-x})\sin y dy = 0$

$\Rightarrow dx + (1 + e^{-x})\tan y dy = 0$

$\Rightarrow \frac{dx}{1 + e^{-x}} + \tan y dy = 0$

Integrating both sides, we get

$$\int \frac{e^x}{1 + e^x} dx + \int \tan y dy = 0$$

$$\Rightarrow \log(1 + e^x) + \log|\sec y| = \log C$$

$$\Rightarrow \sec y(1 + e^x) = C$$

When, $x = 0, y = \frac{\pi}{4}$, we get $C = 2\sqrt{2}$

$\therefore$ Particular solution of the differential equation is, $\sec y(1 + e^x) = 2\sqrt{2}$

6. Solve the differential equation:

$\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$, given that y = 1 when x = 0.

[CBSE 2019]

Solution

We have, $\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$

$\therefore \frac{dy}{dx} = 1 + x^2 + y^2(1 + x^2) = (1 + x^2) \cdot (1 + y^2)$

$\Rightarrow \frac{dy}{1 + y^2} = (1 + x^2)dx$

Integrating both sides, we get $\tan^{-1} y = x + \frac{x^3}{3} + C$ when $x = 0, y = 1$

$\tan^{-1} 1 = 0 + 0 + C \Rightarrow C = \frac{\pi}{4}$

$\therefore \tan^{-1} y = x + \frac{1}{3}x^3 + \frac{\pi}{4}$ is the required solution.

7. Solve the differential equation $(x^2\ -\ yx^2)dy\ +\ (y^2 +\ x^2y^2)dx = 0$, given that y = 1 when x = 1.

[CBSE 2014]

Solution

We have, $(x^2 - yx^2)dy + (y^2 + x^2y^2)dx = 0$

$\Rightarrow x^2(1-y)dy + y^2(1+x^2)dx = 0$

$\Rightarrow \int \frac{(1-y)}{y^2}dy + \int \frac{(1+x^2)}{x^2}dx = 0$

$\Rightarrow \int \left(\frac{1}{y^2} - \frac{1}{y}\right)dy + \int \left(\frac{1}{x^2} + 1\right)dx = 0$

$\Rightarrow -\frac{1}{y} - \log|y| - \frac{1}{x} + x = C$

$\Rightarrow \ -x - xy\log|y| - y + x^2y = C(xy)$

when $x = 1, y = 1$

$\therefore \ -(1) - (1)(1)\log|1| - (1) + (1)^2(1) = C(1)$

$\Rightarrow C = -1$

Equation (i) becomes

$$x^2y = x + xy\log|y| + y - xy$$

8. If y(x) is a solution of the differential equation $\left(\frac{2+\sin x}{1+y}\right)\frac{dy}{dx} = -\cos x$ and $y(O) = 1$, then find the value of $y\left(\frac{\pi}{2}\right)$.

[CBSE 2014C]

Solution

We have, $\left(\frac{2+\sin x}{1+y}\right)\frac{dy}{dx} = -\cos x$

$\Rightarrow \frac{dy}{1+y} = -\frac{\cos x}{2+\sin x}dx$

...(i)

Integrating both sides, we get

$\log(y+1) = -\log(2+\sin x) + \log C$

$\Rightarrow \log(y+1) = \log\frac{C}{2+\sin x}$

$\Rightarrow y + 1 = C/(2+\sin x) \Rightarrow (y+1)(2+\sin x) = C$

Given : $y(0) = 1 \Leftrightarrow x = 0, y = 1$

$\therefore \ (1+1)\cdot(2+\sin 0) = C \Rightarrow C = 4$

$\therefore \ (y+1)(2+\sin x) = 4$

$\Rightarrow y = \frac{4}{2+\sin x} - 1$

Put $x = \frac{\pi}{2}$ in (i), $y\left(\frac{\pi}{2}\right) = \frac{4}{2+1} - 1 = \frac{1}{3}$.

9. Find the particular solution of the differential equation $x\,dy/dx = y - x\tan\left(\frac{y}{x}\right)$, given that $y = \frac{\pi}{4}$ at $x = 1$.

[CBSE 2020]

Solution

We have, $x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \tan\left(\frac{y}{x}\right)$,

This is a homogeneous differential equation.

Now, put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore\ v + x\frac{dv}{dx} = v - \tan v$

$\Rightarrow\ x\frac{dv}{dx} = -\tan v \Rightarrow \frac{dv}{\tan v} = -\frac{dx}{x} \Rightarrow \cot v dv + \frac{dx}{x} = 0$

Integrating both sides,we get

$\log|\sin v| + \log x = \log C$

$\Rightarrow x\sin v = C \Rightarrow x\sin\left(\frac{y}{x}\right) = C$

When $x = 1, y = \frac{\pi}{4}$, we get $1 \cdot \sin\left(\frac{\pi}{4}\right) = C \Rightarrow C = \frac{1}{\sqrt{2}}$

So, $x\sin\left(\frac{y}{x}\right) = \frac{1}{\sqrt{2}}$ is the required particular solution.

10. Solve the following differential equation:

$$\left[y - x\cos\left(\frac{y}{x}\right)\right]dy + \left[y\cos\left(\frac{y}{x}\right) - 2x\sin\left(\frac{y}{x}\right)\right]dx = 0$$

[CBSE 2015]

Solution

We have,

$$\left[y - x\cos\left(\frac{y}{x}\right)\right]dy + \left[y\cos\left(\frac{y}{x}\right) - 2x\sin\left(\frac{y}{x}\right)\right]dx = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x\sin\left(\frac{y}{x}\right) - y\cos\left(\frac{y}{x}\right)}{y - x\cos\left(\frac{y}{x}\right)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2\sin\left(\frac{y}{x}\right) - \frac{y}{x}\cos\left(\frac{y}{x}\right)}{\frac{y}{x} - \cos\left(\frac{y}{x}\right)}$$

This is a homogeneous differential equation.

$\therefore$ Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ Equation (i) becomes,

$$v + x\frac{dv}{dx} = \frac{2\sin v - v\cos v}{v - \cos v} \Rightarrow x\frac{dv}{dx} = \frac{2\sin v - v\cos v}{v - \cos v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2\sin v - v^2}{v - \cos v} \Rightarrow \frac{v - \cos v}{2\sin v - v^2}dv = \frac{dx}{x}$$

$$\Rightarrow \frac{-1}{2}\frac{(2\cos v - 2v)}{2\sin v - v^2}dv = \frac{dx}{x}$$

Integrating both sides, we get

$$\frac{-1}{2}\log(2\sin v - v^2) = \log x + C_1$$

$$\Rightarrow \log x^2 + 2C_1 + \log\left(2\sin\frac{y}{x} - \frac{y^2}{x^2}\right) = 0$$

$$\Rightarrow \log\left[x^2\left(2\sin\frac{y}{x} - \frac{y^2}{x^2}\right)\right] = -2C_1$$

$\Rightarrow 2x^2\sin\frac{y}{x} - y^2 = e^{-2C_1} = C$ (say)

which is the required solution.

11. Find the integrating factor of the differential equation $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$.

[CBSE 2015, 2015C]

Solution

We have, $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$

or $\frac{dy}{dx} + \frac{1}{\sqrt{x}}y = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{1}{\sqrt{x}}, Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

$$\therefore \text{ I.F. } = e^{\int P dx} \Rightarrow \text{ I.F. } = e^{\int \frac{1}{\sqrt{x}}dx} = e^{2\sqrt{x}}$$

12. Write the integrating factor of the following differential equation: $(1 + y^2) + (2xy - \cot y)\frac{dy}{dx} = 0$

[CBSE 2015]

Solution

The given differential equation is

$(1+y^2)+(2xy-\cot y)\frac{dy}{dx}=0$

$\Rightarrow (1+y^2)\frac{dx}{dy}+2xy-\cot y=0 \Rightarrow \frac{dx}{dy}+\frac{2y}{1+y^2}\cdot x=\frac{\cot y}{1+y^2}$

This is a linear differential equation of the form $\frac{dx}{dy}+Px=Q$, where, $P=\frac{2y}{1+y^2}$ and $Q=\frac{\cot y}{1+y^2}$

$$\therefore \text{ I.F. } = e^{\int P dy} = e^{\int \frac{2y}{1+y^2}dy} = e^{\log(1+y^2)} = 1+y^2.$$

13. Find the particular solution of the differential equation $(\tan^{-1} y - x)dy = (1+y^2)dx$, given that x = 1 when y = 0.

[CBSE 2015]

Solution

We have, $(\tan^{-1} y - x)dy = (1+y^2)dx$

$\Rightarrow \frac{dx}{dy}=\frac{\tan^{-1} y - x}{1+y^2} \Rightarrow \frac{dx}{dy}+\frac{1}{1+y^2}\cdot x=\frac{\tan^{-1} y}{1+y^2}$

Differential Equations

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This is a linear differential equation of the form

$\frac{dx}{dy}+Px=Q$, where $P=\frac{1}{1+y^2}$ and $Q=\frac{\tan^{-1} y}{1+y^2}$

$$\text{I.F. } = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1} y}$$

∴ Required solution is,

$$x\cdot e^{\tan^{-1} y} = \int \frac{e^{\tan^{-1} y}\cdot \tan^{-1} y}{1+y^2}dy + C$$

Put $\tan^{-1} y = t \Rightarrow \left(\frac{1}{1+y^2}\right)dy = dt$

∴ (i) becomes,

$$x\cdot e^{\tan^{-1} y} = \int e^t \cdot t dt + C$$

$$\Rightarrow x\cdot e^{\tan^{-1} y} = t\cdot e^t - \int 1\cdot e^t dt + C$$

$$\Rightarrow x\cdot e^{\tan^{-1} y} = t\cdot e^t - e^t + C$$

$$\Rightarrow x\cdot e^{\tan^{-1} y} = \tan^{-1} y\cdot e^{\tan^{-1} y} - e^{\tan^{-1} y} + C$$

$$\Rightarrow x = \tan^{-1} y - 1 + Ce^{\tan^{-1} y}$$

We get the solution as

$$x = \tan^{-1} y - 1 + Ce^{-\tan^{-1} y}$$

Now, putting $x = 1, y = 0$ in (i), we get

$$1 = \tan^{-1} 0 - 1 + Ce^{-\tan^{-1} 0} \Rightarrow C = 2$$

So, required particular solution is $x = \tan^{-1} y - 1 + 2e^{-\tan^{-1} y}$

14. Find the particular solution of the differential equation $x\frac{dy}{dx} + y - x + xy\cot x = 0; x \neq 0$, given that when $x = \frac{\pi}{2}, y = 0$.

[CBSE 2015C]

Solution

We have, $x\frac{dy}{dx} + y - x + xy\cot x = 0, (x \neq 0)$

$\Rightarrow x\frac{dy}{dx} + (1 + x\cot x) \cdot y = x$

$\Rightarrow \frac{dy}{dx} + \frac{1 + x\cot x}{x} \cdot y = 1$

This is linear D.E. of the form $\frac{dy}{dx} + Py = Q$

where $P = \frac{1+x\cot x}{x} = \frac{1}{x} + \cot x$ and $Q = 1$

$\therefore$ Now I.F. $= e^{\int P dx} = e^{\log(x\sin x)} = x\sin x$

$\therefore$ The solution. of (i) is $y \cdot x\sin x = \int 1 \cdot x\sin x dx + C$

$= x(-\cos x) + \int 1 \cdot \cos x dx + C.$

$\Rightarrow xy\sin x = -x\cos x + \sin x + C$

The required solution

$y \cdot x\sin x = x(-\cos x) + \sin x + C$

Putting $x = \frac{\pi}{2}, y = 0$ in (i), we get

$\therefore 0 = -\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2} + C \Rightarrow C = -1$

$xy\sin x = \sin x - x\cos x - 1$ is the required particular solution.

15. The sum of the order and the degree of the differential equation $\frac{d}{dx}\left(\left(\frac{dy}{dx}\right)^3\right)$ is

(a) 2
(b) 3
(c) 5
(d) 0

[CBSE 2023]

Solution

(b) : [There is error in question, the given differential equation should be $\frac{d}{dx}\left(\frac{dy}{dx}\right)^3 = 0$.]

The given differential equation is,

$$\frac{d}{dx}\left(\left(\frac{dy}{dx}\right)^3\right) = 0 \Rightarrow 3\left(\frac{dy}{dx}\right)^2\left(\frac{d^2y}{dx^2}\right) = 0$$

$\therefore$ Order $= 2$ and degree $= 1$. So, required sum $= 2 + 1 = 3$

16. The order and the degree of the differential equation $\left(1 + 3\frac{dy}{dx}\right)^2 = 4\frac{d^3y}{dx^3}$ respectively are

(a) $1, \frac{2}{3}$
(b) 3,1
(c) 3,3
(d) 1,2

[CBSE 2023]

Solution

. (b) : We have, $\left(1 + 3\frac{dy}{dx}\right)^2 = 4\frac{d^3y}{dx^3}$

Here, order $= 3$ as highest order derivative is $\frac{d^3y}{dx^3}$.

And degree $= 1$, as power of highest order derivative i.e., $\frac{d^3y}{dx^3}$ is 1 .

17. Solve the differential equation given by

$$xdy - ydx - \sqrt{x^2 + y^2}dx = 0.$$

[CBSE 2023]

Solution

Given differential equation is,

$xdy - ydx - \sqrt{x^2 + y^2}dx = 0$

$\Rightarrow xdy = \left(y + \sqrt{x^2 + y^2}\right)dx$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2} \#(i)$$

Put $\frac{y}{x} = V \Rightarrow y = Vx \Rightarrow \frac{dy}{dx} = V + x\frac{dV}{dx}$

Putting the value of $\frac{dy}{dx}$ and $\frac{y}{x}$ in (i), we get

$\Rightarrow V + x\frac{dV}{dx} = V + \sqrt{1 + V^2} \Rightarrow x\frac{dV}{dx} = \sqrt{1 + V^2}$

$\Rightarrow \frac{d\,V}{\sqrt{1+V^2}} = \frac{dx}{x}$

Integrating both sides, we get

$\log\left(V + \sqrt{1 + V^2}\right) = \log x + \log C$

$\Rightarrow \log\left(\frac{y}{x}+\sqrt{1+\left(\frac{y}{x}\right)^2}\right)=\log x+\log C$

$\Rightarrow \log\left(y+\sqrt{x^2+y^2}\right)-\log x=\log x+\log c$

$\Rightarrow \log\left(y+\sqrt{x^2+y^2}\right)=2\log x+\log C \Rightarrow Cx^2=y+\sqrt{x^2+y^2}$,

which is the required solution of the differential equation.

18. Find the particular solution of the differential equation $\frac{dy}{dx}=\frac{x+y}{x}, y(1)=0$.

[CBSE 2023]

Solution

Given differential equation is $\frac{dy}{dx}=\frac{x+y}{x} \Rightarrow \frac{dy}{dx}-\frac{y}{x}=1$ Clearly, it is a linear differential equation of the form $\frac{dy}{dx}+Py=Q$, where $P=\frac{-1}{x}, Q=1$

I.F. $=e^{\int \frac{-1}{x}dx}=e^{-\log x}=\frac{1}{x}$

$\therefore$ Solution is given by $y\cdot\frac{1}{x}=\int 1\cdot\frac{1}{x}dx+C$

$$\Rightarrow \frac{y}{x}=\log x+C \#(i)$$

We have, $y(1)=0$ i.e., when $x=1, y=0$

$\therefore\ 0=0+C \Rightarrow C=0$

$\therefore$ From (i) $\frac{y}{x}=\log x \Rightarrow y=x\log x$

19. Find the general solution of the differential equation:

$$ydx=(x+2y^2)dy$$

[CBSE 2024]

Solution

The given differential equation is $ydx=(x+2y^2)dy$

$\Rightarrow\ ydx=xdy+2y^2dy \Rightarrow ydx-xdy=2y^2dy$

$\Rightarrow \frac{ydx-xdy}{y^2}=2dy \Rightarrow d\left(\frac{x}{y}\right)=2dy$

Integrating both sides, we get

$\int d\left(\frac{x}{y}\right)=2\int dy \Rightarrow \frac{x}{y}=2y+C$, which is the required general solution.

20. Find the particular solution of the differential equation, $\frac{dy}{dx}-2xy=3x^2e^{x^2}; y(0)=5$.

[CBSE 2024]

Solution

Given, $\frac{dy}{dx}-2xy=3x^2e^{x^2}$

which is in the form of $\frac{dy}{dx}+Py=Q$,

where $P=-2x, Q=3x^2e^{x^2}$

$\therefore$ I.F. $=e^{\int Pdx}=e^{\int(-2x)dx}=e^{-x^2}$

$\therefore$ Required solution of given differential equation is

$y\times I.\text{F.}=\int(Q\times I.\text{F.})dx+C \Rightarrow y\left(e^{-x^2}\right)=\int\left(3x^2e^{x^2}\times e^{-x^2}\right)dx+C$

$\Rightarrow y\left(e^{-x^2}\right)=3\int x^2dx+C \Rightarrow y\left(e^{-x^2}\right)=x^3+C$

Now, y(0) = 5 i.e., when x = 0, then y = 5 $\therefore 5(e^{-0})=0+C \Rightarrow C=5$

$\therefore$ Particular solution is $y\left(e^{-x^2}\right)=x^3+5$

21. Solve the following differential equation:

$$x^2dy + y(x+y)dx = 0.$$

[CBSE 2024]

Solution

Given, differential equation is $x^2dy + y(x+y)dx = 0$

$$\Rightarrow x^2dy = (-yx - y^2)dx \Rightarrow \frac{dy}{dx} = -\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2 \#(i)$$

Which is a homogeneous differential equation as

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

On putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = -v - v^2 \Rightarrow x\frac{dv}{dx} = -2v - v^2 \Rightarrow \frac{1}{2v+v^2}dv = -\frac{1}{x}dx \Rightarrow \frac{1}{v(2+v)}dv = -\frac{1}{x}dx$$

$$\Rightarrow \frac{2}{2v(2+v)} = -\frac{1}{x}dx \Rightarrow \frac{1}{2}\left(\frac{1}{v} - \frac{1}{v+2}\right)dv = -\frac{1}{x}dx \Rightarrow \frac{1}{2}\int \frac{1}{v}dv - \frac{1}{2}\int \frac{1}{v+2}dv$$

$$= -\int \frac{1}{x}dx \Rightarrow \frac{1}{2}\log|v| - \frac{1}{2}\log|v+2| = -\log|x| + \log C \Rightarrow \frac{1}{2}\log\left|\frac{v}{v+2}\right| = \log\left(\frac{C}{x}\right)$$

$$\Rightarrow \log\left|\frac{v}{v+2}\right| = 2\log\left(\frac{C}{x}\right) \Rightarrow \frac{v}{v+2} = \left(\frac{C}{x}\right)^2 \Rightarrow \frac{y/x}{(y/x)+2} = \left(\frac{C}{x}\right)^2 \Rightarrow \frac{y}{y+2x} = \left(\frac{C}{x}\right)^2$$

$\Rightarrow x^2y = c(y+2x)$, which is the required solution.

22. Find the particular solution of the differential equation $\left(xe^{y/x} + y\right)dx = xdy$, given that $y = 1$ when $x = 1$.

Solution

We have, $\left(xe^{y/x} + y\right)dx = xdy$

$$\Rightarrow e^{y/x} + \frac{y}{x} = \frac{dy}{dx} \#(i)$$

Put $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx} \Rightarrow v + x\frac{dv}{dx} = e^v + v$$

[From (i)]

$$\Rightarrow x\frac{dv}{dx} = e^v \Rightarrow \int \frac{dv}{e^v} = \int \frac{dx}{x} \Rightarrow -e^{-v} = \log x + C \Rightarrow -e^{-y/x} = \log x + C$$

At $x = 1, y = 1$, then $-e^{-1} = C; C = \frac{-1}{e}$

$\therefore$ Required solution is $-e^{-y/x} = \log x - \frac{1}{e}$

23. The order and degree of the following differential equation are, respectively :

$$-\frac{d^4y}{dx^4} + 2e^{dy/dx} + y^2 = 0$$

(a) $-4,1$

(b) 4, not defined

(c) 1,1

(d) 4,1

[CBSE 2025]

Solution

(b) : Given differential equation is

$$\frac{-d^4y}{dx^4} + 2e^{dy/dx} + y^2 = 0$$

24. The order of highest order derivative occuring in the given differential equation is 4.
As the given equation is not a polynomial equation in $\frac{dy}{dx}$. Hence, degree is not defined.
The integrating factor of the differential equation $\frac{dy}{dx} + y = \frac{1+y}{x}$ is
(a) xe^x
(b) $\frac{e^x}{x}$
(c) $\frac{x}{e^x}$
(d) xe^x

[CBSE 2025]

Solution

. (b): We have, $\frac{dy}{dx} + y = \frac{1+y}{x} \Rightarrow \frac{dy}{dx} + y - \frac{y}{x} = \frac{1}{x}$
$\Rightarrow \frac{dy}{dx} + y\left(1 - \frac{1}{x}\right) = \frac{1}{x} \Rightarrow \frac{dy}{dx} + y\left(1 - \frac{1}{x}\right) = \frac{1}{x}$,
where $P = 1 - \frac{1}{x}$ and $Q = \frac{1}{x}$
$\therefore$ Integrating factor, I.F. $= e^{\int P dx} = e^{\int \left(1-\frac{1}{x}\right)dx}$

$$= e^{|x - \log x|} = e^x e^{\log x^{-1}} = \frac{e^x}{x}$$

25. Solve the differential equation $2(y+3) - xy\frac{dy}{dx} = 0$; given $y(1) = -2$.

[CBSE 2025]

Solution

We have, $2(y+3) - xy\frac{dy}{dx} = 0$
$\Rightarrow xy\frac{dy}{dx} = 2(y+3) \Rightarrow \frac{y}{y+3}dy = 2\frac{dx}{x}$
Integrating both sides w.r.t. x, we get
$\int \left(\frac{y+3}{y+3} - \frac{3}{y+3}\right)dy = 2\int \frac{dx}{x} \Rightarrow \int 1 \cdot dy - 3\int \frac{dy}{y+3} = 2\int \frac{dx}{x} \Rightarrow y - 3\log(y+3) = 2\log x + C$
Given, $y(1) = -2$ i.e., when $x = 1, y = -2$, then

$$-2 - 3\log 1 = 2\log 1 + C \Rightarrow C = -2$$

$\therefore\ y - 3\log(y+3) = 2\log x - 2 \Rightarrow y + 2 = 2\log x + 3\log(y+3) \Rightarrow y + 2 = \log x^2(y+3)^3$
$\Rightarrow e^{y+2} = x^2(y+3)^3$, which is required solution of given differential equation.

26. Solve the differential equation :
$x^2ydx - (x^3 + y^3)dy = 0$.

[CBSE 2025]

Solution

We have, $x^2ydx - (x^3 + y^3)dy = 0$
$\Rightarrow \frac{dy}{dx} = \frac{x^2y}{x^3 + y^3}$
Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$
Now, $\frac{dy}{dx} = \frac{x^2(vx)}{x^3+(vx)^3} = \frac{x^3v}{x^3+v^3x^3} = \frac{v}{1+v^3} \Rightarrow v + x\frac{dv}{dx} = \frac{v}{1+v^3}$
$\Rightarrow x\frac{dv}{dx} = \frac{v}{1+v^3} - v \Rightarrow x\frac{dv}{dx} = \frac{-v^4}{1+v^3} \Rightarrow \frac{1+v^3}{v^4}dv = \frac{-dx}{x}$
Integrating on both sides, we get

$$\int \left(\frac{1}{v^4}+\frac{1}{v}\right)dv = -\int \frac{dx}{x} \Rightarrow \frac{v^{-3}}{-3} + \ln|v| = -\ln|x| + \ln|C_1| \Rightarrow \frac{-x^3}{3y^3} + \ln\left|\frac{y}{x}\right| = -\ln|x| + \ln|C_1|$$

$$\Rightarrow \frac{-x^3}{3y^3} + \ln|y| - \ln|x| + \ln C = -\ln|x| [\because \ln|C| = -\ln|C_1|] \Rightarrow \frac{x^3}{3y^3} = \ln|yC|$$

Vector Previous Years Questions

1. Write a unit vector in the direction of vector $\overrightarrow{PQ}$, where P and Q are the points (1, 3, 0) and (4, 5, 6) respectively.

[CBSE 2014]

Solution

We have, $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= (4\hat{\imath} + 5\hat{\jmath} + 6\hat{k}) - (\hat{\imath} + 3\hat{\jmath}) = 3\hat{\imath} + 2\hat{\jmath} + 6\hat{k}$

Requiredunit vector $= \frac{3\hat{\imath}+2\hat{\jmath}+6\hat{k}}{\sqrt{3^2+2^2+6^2}} = \frac{3\hat{\imath}+2\hat{\jmath}+6\hat{k}}{7}$

2. Write a unit vector in the direction of the sum of the vectors $\vec{a} = 2\hat{\imath} - 5\hat{k}$ and $\vec{b} = 2\hat{\imath} - 4\hat{\jmath} + 5\hat{k}$.

[CBSE 2014]

Solution

We have, $\vec{a} = 2\hat{\imath} - 5\hat{k}$ and $\vec{b} = 2\hat{\imath} + \hat{\jmath} - 7\hat{k}$

Sum of the given vectors is

$\vec{c} = \vec{a} + \vec{b} = (2+2)\hat{\imath} + (0-4)\hat{\jmath} + (-5+5)\hat{k} = 4\hat{\imath} - 4\hat{\jmath}$

and $|\vec{c}| = \sqrt{(4)^2 + (-4)^2} = \sqrt{32} = 4\sqrt{2}$

$\therefore$ Unit vector, $\hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{4\hat{\imath}-4\hat{\jmath}}{4\sqrt{2}}$

3. Find a vector in the direction of vector $2\hat{\imath} - 3\hat{\jmath} + 6\hat{k}$ which has magnitude 21 units.

[CBSE 2014]

Solution

Let $\vec{a} = 2\hat{\imath} - 3\hat{\jmath} + 6\hat{k}$.

The vector in the direction of $\vec{a}$ with a magnitude of $21 = 21$ units $\times \hat{a}$

$\therefore$ Required vector $= 21 \times \frac{2\hat{\imath}-3\hat{\jmath}+6\hat{k}}{\sqrt{2^2+(-3)^2+6^2}}$

$$= 21 \times \frac{2\hat{\imath} - 3\hat{\jmath} + 6\hat{k}}{7} = 6\hat{\imath} - 9\hat{\jmath} + 18\hat{k}$$

4. Write the value of cosine of the angle which the vector $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$ makes with y-axis.

[CBSE 2014C]

Solution

Let θ be the angle between the vector $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$ and y-axisi.e., $\vec{b} = \hat{\jmath}$

$$\therefore \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

$$= \frac{(\hat{\imath}+\hat{\jmath}+\hat{k})\hat{\jmath}}{|\hat{\imath}+\hat{\jmath}+\hat{k}||\hat{\jmath}|} = \frac{1}{\sqrt{1^2+1^2+1^2}\sqrt{1^2}} = \frac{1}{\sqrt{3}}$$

5. Find a vector $\vec{a}$ of magnitude $5\sqrt{2}$, making an angle of $\frac{\pi}{4}$ with x-axis, $\frac{\pi}{2}$ with y-axis and an acute angle θ with z-axis.

[CBSE 2014]

Solution

Here, $l = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}, m = \cos\frac{\pi}{2} = 0,$

$n = \cos\theta$

Since, $l^2 + m^2 + n^2 = 1$

$\Rightarrow \frac{1}{2} + 0 + \cos^2\theta = 1 \Rightarrow \cos^2\theta = 1 - \frac{1}{2} = \frac{1}{2}$

$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

$\therefore$ The vector of magnitude $5\sqrt{2}$ is

$$\vec{a} = 5\sqrt{2}(l\hat{\imath} + m\hat{\jmath} + n\hat{k})$$

$$= 5\sqrt{2}\left(\frac{1}{\sqrt{2}}\hat{\imath} + 0\hat{\jmath} + \frac{1}{\sqrt{2}}\hat{k}\right) = 5(\hat{\imath}+\hat{k})$$

[$\because \theta$ is an acute angle]

6. Find the position vector of a point which divides the join of points with position vectors $\vec{a} - 2\vec{b}$ and $2\vec{a} + \vec{b}$ externally in the ratio 2 : 1.

[CBSE 2016]

Solution

Required position vector

$$= \frac{2\cdot(2\vec{a}+\vec{b}) - 1(\vec{a}-2\vec{b})}{2-1} = \frac{4\vec{a}+2\vec{b}-\vec{a}+2\vec{b}}{1} = 3\vec{a}+4\vec{b}$$

7. Find the magnitude of each of the two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude such that the angle between them is 60° and their scalar product is $\frac{9}{2}$.

[CBSE 2018]

Solution

Given, $|\vec{a}| = |\vec{b}|, \theta = 60°$ and $\vec{a}\cdot\vec{b} = \frac{9}{2}$

Now, $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

$\Rightarrow \cos 60° = \frac{9/2}{|\vec{a}|^2} \Rightarrow \frac{1}{2} = \frac{9/2}{|\vec{a}|^2}$

$\Rightarrow |\vec{a}|^2 = 9 \Rightarrow |\vec{a}| = 3 \therefore |\vec{a}| = |\vec{b}| = 3$

8. The scalar product of the vector $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$ with a unit vector along the sum of vectors $\vec{b} = 2\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$ and $\vec{c} = \lambda\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ is equal to one. Find the value of λ and hence find the unit vector along $\vec{b} + \vec{c}$.

[CBSE 2014]

Solution

Here, $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}, \vec{b} = 2\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$ and

$$\vec{c} = \lambda\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$$

$$\Rightarrow \vec{b} + \vec{c} = (2 + \lambda)\hat{\imath} + 6\hat{\jmath} - 2\hat{k}$$

The unit vector along $\vec{b} + \vec{c}$ is $\vec{p} = \frac{\vec{b}+\vec{c}}{|\vec{b}+\vec{c}|}$

$$= \frac{(2 + \lambda)\hat{\imath} + 6\hat{\jmath} - 2\hat{k}}{\sqrt{(2 + \lambda)^2 + 6^2 + (-2)^2}} = \frac{(2 + \lambda)\hat{\imath} + 6\hat{\jmath} - 2\hat{k}}{\sqrt{\lambda^2 + 4\lambda + 44}}$$

Also, $\vec{a} \cdot \vec{p} = 1$ (Given)

$$\Rightarrow \frac{(2 + \lambda) + 6 - 2}{\sqrt{\lambda^2 + 4\lambda + 44}} = 1$$

$$\Rightarrow \sqrt{\lambda^2 + 4\lambda + 44} = \lambda + 6$$

$$\Rightarrow \lambda^2 + 4\lambda + 44 = \lambda^2 + 12\lambda + 36$$

$$\Rightarrow 8\lambda = 8 \Rightarrow \lambda = 1$$

∴ The required unit vector

$$\vec{p} = \frac{(2 + 1)\hat{\imath} + 6\hat{\jmath} - 2\hat{k}}{\sqrt{1 + 4 + 44}} = \frac{1}{7}(3\hat{\imath} + 6\hat{\jmath} - 2\hat{k}).$$

9. Find the value of 'p' for which the vectors $3\hat{\imath} + 2\hat{\jmath} + 9\hat{k}$ and $\hat{\imath} - 2p\hat{\jmath} + 3\hat{k}$ are parallel.

[CBSE 2014]

Solution

. Let $\vec{a} = 3\hat{\imath} + 2\hat{\jmath} + 9\hat{k}$ and $\vec{b} = \hat{\imath} - 2p\hat{\jmath} + 3\hat{k}$

For $\vec{a}$ and $\vec{b}$ to be parallel, $\vec{b} = \lambda\vec{a}$.

$\Rightarrow \hat{\imath} - 2p\hat{\jmath} + 3\hat{k} = \lambda(3\hat{\imath} + 2\hat{\jmath} + 9\hat{k}) = 3\lambda\hat{\imath} + 2\lambda\hat{\jmath} + 9\lambda\hat{k}$

$\Rightarrow 1 = 3\lambda, -2p = 2\lambda, 3 = 9\lambda$

$\Rightarrow \lambda = \frac{1}{3}$ and $p = -\lambda = -\frac{1}{3}$

10. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a} + \vec{b}| = |\vec{a}|$, then prove that vector $2\vec{a} + \vec{b}$ is perpendicular to $\vec{b}$

[CBSE 2013]

Solution

Here $|\vec{a}+\vec{b}|=|\vec{a}|$

$\Rightarrow |\vec{a}+\vec{b}|^2=|\vec{a}|^2$

$\Rightarrow (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=\vec{a}\cdot\vec{a}$

$\Rightarrow \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}=\vec{a}\cdot\vec{a}$

$\Rightarrow 2\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}=0$ [$\because \vec{b}\cdot\vec{a}=\vec{a}\cdot\vec{b}$]

$\Rightarrow (2\vec{a}+\vec{b})\cdot\vec{b}=0 \Rightarrow (2\vec{a}+\vec{b})\perp\vec{b}$

11. Write the projection of $\vec{b}+\vec{c}$ on $\vec{a}$, where $\vec{a}=2\hat{\imath}-2\hat{\jmath}+\hat{k}, \vec{b}=\hat{\imath}+2\hat{\jmath}-2\hat{k}$ and $2\hat{\imath}-\hat{\jmath}+4\hat{k}$.

[CBSE 2013C]

Solution

Here, $\vec{a}=2\hat{\imath}-2\hat{\jmath}+\hat{k}$

$\vec{b}=\hat{\imath}+2\hat{\jmath}-2\hat{k}$ and $\vec{c}=2\hat{\imath}-\hat{\jmath}+4\hat{k}$

$\Rightarrow \vec{b}+\vec{c}=3\hat{\imath}+\hat{\jmath}+2\hat{k}$

$\therefore$ Projection of $\vec{b}+\vec{c}$ on $\vec{a}=\frac{(\vec{b}+\vec{c})\cdot\vec{a}}{|\vec{a}|}$

$$=\frac{(3\hat{\imath}+\hat{\jmath}+2\hat{k})\cdot(2\hat{\imath}-2\hat{\jmath}+\hat{k})}{|2\hat{\imath}-2\hat{\jmath}+\hat{k}|}$$

$$=\frac{3\times2+1\times(-2)+2\times1}{\sqrt{2^2+(-2)^2+1^2}}=\frac{6}{3}=2$$

Based on $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$

12. If $\vec{a},\vec{b},\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, then write the value of $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$.

[CBSE 2016]

Solution

We have $\vec{a},\vec{b},\vec{c}$ are unit vectors.

Therefore, $|\vec{a}|=1, |\vec{b}|=1$ and $|\vec{c}|=1$

Also, $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

(given)

$\Rightarrow |\vec{a}+\vec{b}+\vec{c}|^2=0$

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$

$\Rightarrow 1+1+1+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$

$\Rightarrow 3+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$

$\Rightarrow (\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=-\frac{3}{2}$

13. If $\vec{a}+\vec{b}+\vec{c}=0$ and $|\vec{a}|=3, |\vec{b}|=5, |\vec{c}|=7$, then find the value of $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$.

[CBSE 2019-20]

Solution

We have,

$(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})=0 \quad [\because \vec{a}+\vec{b}+\vec{c}=0]$

$\Rightarrow \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}+\vec{c}\cdot\vec{c}=0$ (1)

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$

$\Rightarrow 3^2+5^2+7^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$

$\Rightarrow 2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=-(9+25+49)$

$\Rightarrow (\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=-\frac{83}{2}$

14. If $\vec{a},\vec{b}$ and $\vec{c}$ are three vectors such that each one is perpendicular to the vector obtained by sum of the other two and $|\vec{a}|=3, |\vec{b}|=4$ and $|\vec{c}|=5$, then prove that $|\vec{a}+\vec{b}+\vec{c}|=5\sqrt{2}$.

[CBSE 2013C]

Solution

Given, $|\vec{a}|=3, |\vec{b}|=4, |\vec{c}|=5$

and $\vec{a}\cdot(\vec{b}+\vec{c})=0, \vec{b}\cdot(\vec{c}+\vec{a})=0, \vec{c}\cdot(\vec{a}+\vec{b})=0$

$\therefore \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}+\vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}$

$=0+0+0=0$

$\Rightarrow 2(\vec{a}\cdot\vec{b})+2(\vec{b}\cdot\vec{c})+2(\vec{c}\cdot\vec{a})=0$

Now $|\vec{a}+\vec{b}+\vec{c}|^2=(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$

$=(\vec{a}+\vec{b}+\vec{c})\cdot\vec{a}+(\vec{a}+\vec{b}+\vec{c})\cdot\vec{b}+(\vec{a}+\vec{b}+\vec{c})\cdot\vec{c}$

$=\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}+\vec{c}\cdot\vec{b}+\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{c}$

$=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b})+2(\vec{b}\cdot\vec{c})+2(\vec{c}\cdot\vec{a})$

$=3^2+4^2+5^2+0$

$=50$

$\therefore |\vec{a}+\vec{b}+\vec{c}|=5\sqrt{2}$.

15. If $\vec{a}$ and $\vec{b}$ are unit vectors, then what is the angle between $\vec{a}$ and $\vec{b}$ so that $\sqrt{2}\,\vec{a}-\vec{b}$ is a unit vector?

[CBSE 2015C]

Solution

Let θ be the angle between the unit vectors $\vec{a}$ and $\vec{b}$.

$$\therefore \cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\vec{a}\cdot\vec{b}\ (\because |\vec{a}|=1=|\vec{b}|)$$

Now, $1=|\sqrt{2}\vec{a}-\vec{b}|$

$$\Rightarrow 1 = |\sqrt{2}\vec{a} - \vec{b}|^2 = (\sqrt{2}\vec{a} - \vec{b}) \cdot ($$
$$= 2|\vec{a}|^2 - \sqrt{2}\vec{a} \cdot \vec{b} - \vec{b} \cdot \sqrt{2}\vec{a} + |\vec{b}|^2$$
$$= 2 - 2\sqrt{2}\vec{a} \cdot \vec{b} + 1$$
$$= 3 - 2\sqrt{2}\vec{a} \cdot \vec{b}$$
$$\Rightarrow \ \vec{a} \cdot \vec{b} = \frac{1}{\sqrt{2}} \Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$$
$$\therefore \ \theta = \pi/4$$
$$(\because \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a})$$

16. Vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $|\vec{a}| = 3, \left|\vec{b}\right| = 5$ and $|\vec{c}| = 7$. Find the angle between $\vec{a}$ and $\vec{b}$.

[CBSE 2014]

Solution

Given $\vec{a} + \vec{b} + \vec{c} = 0$ and $|\vec{a}| = 3, |\vec{b}| = 5, |\vec{c}| = 7$

We have $\vec{a} + \vec{b} + \vec{c} = 0$

$$\Rightarrow \vec{a} + \vec{b} = -\vec{c} \Rightarrow |\vec{a} + \vec{b}|^2 = |-\vec{c}|^2$$
$$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2(\vec{a} \cdot \vec{b}) = |\vec{c}|^2$$
$$\Rightarrow 9 + 25 + 2|\vec{a}||\vec{b}|\cos\theta = 49$$
$$\Rightarrow 2 \times 3 \times 5 \times \cos\theta = 49 - 34 = 15$$
$$\Rightarrow \cos\theta = \frac{15}{30} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3} = 60°$$

17. Using vectors, find the area of the triangle ABC with vertices A(1, 2, 3), B(2, -1, 4) and C(4, 5, -1)

[CBSE 2020, 2013]

Solution

Given, $\triangle ABC$ with vertices

$$A(1,2,3) = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}, B(2,-1,4) = 2\hat{\imath} - \hat{\jmath} + 4\hat{k},$$
$$C(4,5,-1) = 4\hat{\imath} + 5\hat{\jmath} - \hat{k}$$
Now $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (2\hat{\imath} - \hat{\jmath} + 4\hat{k}) - (\hat{\imath} + 2\hat{\jmath} + 3\hat{k})$
$= \hat{\imath} - 3\hat{\jmath} + \hat{k}$
$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (4\hat{\imath} + 5\hat{\jmath} - \hat{k}) - (\hat{\imath} + 2\hat{\jmath} + 3\hat{k})$
$= 3\hat{\imath} + 3\hat{\jmath} - 4\hat{k}.$

Now $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (2\hat{\imath} - \hat{\jmath} + 4\hat{k}) - (\hat{\imath} + 2\hat{\jmath} + 3\hat{k})$
$= \hat{\imath} - 3\hat{\jmath} + \hat{k}.$
$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (4\hat{\imath} + 5\hat{\jmath} - \hat{k}) - (\hat{\imath} + 2\hat{\jmath} + 3\hat{k})$
$= 3\hat{\imath} + 3\hat{\jmath} - 4\hat{k}.$

$\therefore (\overrightarrow{AB} \times \overrightarrow{AC}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -3 & 1 \\ 3 & 3 & -4 \end{vmatrix} = 9\hat{\imath} + 7\hat{\jmath} + 12$

Hence, area of $\triangle ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

$= \frac{1}{2}|9\hat{\imath} + 7\hat{\jmath} + 12\hat{k}|$

$= \frac{1}{2}\sqrt{9^2 + 7^2 + 12^2} = \frac{1}{2}\sqrt{81 + 49 + 144}$

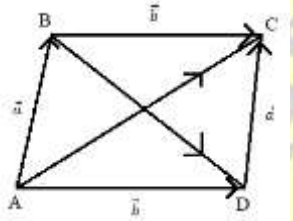
$= \frac{1}{2}\sqrt{274}$ sq. units

18. The two adjacent sides of a parallelogram are $2\hat{\imath} - 4\hat{\jmath} - 5\hat{k}$ and $2\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$. Find the two unit vectors parallel to its diagonals vectors, find the area of the parallelogram.

[CBSE 2016]

Solution

Let $\vec{a} = 2\hat{\imath} - 4\hat{\jmath} - 5\hat{k}$ and $\vec{b} = 2\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$



Then diagonal $\overrightarrow{AC}$ of the parallelogram is

$\vec{p} = \vec{a} + \vec{b}$

$= 2\hat{\imath} - 4\hat{\jmath} - 5\hat{k} + 2\hat{\imath} + 2\hat{\jmath} + 3\hat{k} = 4\hat{\imath} - 2\hat{\jmath} - 2\hat{k}$

Therefore, unit vector parallel to it is

$\frac{\vec{p}}{|\vec{p}|} = \frac{4\hat{\imath} - 2\hat{\jmath} - 2\hat{k}}{\sqrt{16 + 4 + 4}} = \frac{2\hat{\imath} - \hat{\jmath} - \hat{k}}{\sqrt{6}}$

Now, diagonal $\overrightarrow{BD}$ of the parallelogram is

$$\vec{p}' = \vec{b} - \vec{a} = 2\hat{\imath} + 2\hat{\jmath} + 3\hat{k} - 2\hat{\imath} + 4\hat{\jmath} + 5\hat{k} = 6\hat{\jmath} + 8\hat{k}$$

Therefore, unit vector parallel to it is

$$\frac{\vec{p}'}{|\vec{p}'|} = \frac{6\hat{\jmath} + 8\hat{k}}{\sqrt{36 + 64}} = \frac{6\hat{\jmath} + 8\hat{k}}{10} = \frac{3\hat{\jmath} + 4\hat{k}}{5}$$

Now, $\vec{p} \times \vec{p}' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & -2 & -2 \\ 0 & 6 & 8 \end{vmatrix}$

$$= \hat{\imath}(-16 + 12) - \hat{\jmath}(32 - 0) + \hat{k}(24 - 0)$$

$$= -4\hat{\imath} - 32\hat{\jmath} + 24\hat{k}$$

$\therefore$ Area of parallelogram $= \frac{|\vec{p} \times \vec{p}'|}{2}$

$$= \frac{\sqrt{16 + 1024 + 576}}{2} = 2\sqrt{101} \text{ sq. units.}$$

19. Let $\vec{a} = 4\hat{\imath} + 5\hat{\jmath} - \hat{k}, \vec{b} = \hat{\imath} - 4\hat{\jmath} + 5\hat{k}$ and $\vec{c} = 3\hat{\imath} + \hat{\jmath} - \hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{c}$ and $\vec{b}$ and $\vec{d}.\vec{a} = 21$.

[CBSE 2018]

Solution

Let $\vec{d} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Now, it is given that, $\vec{d}$ is perpendicular to $\vec{b} = \hat{\imath} - 4\hat{\jmath} + 5\hat{k}$ and $\vec{c} = 3\hat{\imath} + \hat{\jmath} - \hat{k}$

$\therefore \vec{d} \cdot \vec{b} = 0$ and $\vec{d} \cdot \vec{c} = 0$

$\Rightarrow x - 4y + 5z = 0$

and $3x + y - z = 0$

Also, $\vec{d} \cdot \vec{a} = 21$, where $\vec{a} = 4\hat{\imath} + 5\hat{\jmath} - \hat{k}$

$\Rightarrow 4x + 5y - z = 21$

Eliminating z from (i) and (ii), we get

$16x + y = 0$

Eliminating z from (ii) and (iii), we get $x + 4y = 21$

Solving (iv) and (v), we get

$$x = \frac{-1}{3}, y = \frac{16}{3}$$

Putting the values of x and y in (i), we get $z = \frac{13}{3} \therefore \vec{d} = \frac{-1}{3}\hat{\imath} + \frac{16}{3}\hat{\jmath} + \frac{13}{3}\hat{k}$ is the required vector.

20. If the sum of two unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$.

[CBSE 2019]

Solution

Given, $\hat{a} + \hat{b} = \hat{c}$

$$\Rightarrow (\hat{a} + \hat{b}) \cdot (\hat{a} + \hat{b}) = \hat{c} \cdot \hat{c}$$
$$\Rightarrow \hat{a} \cdot \hat{a} + \hat{a} \cdot \hat{b} + \hat{b} \cdot \hat{a} + \hat{b} \cdot \hat{b} = \hat{c} \cdot \hat{c}$$
$$\Rightarrow 1 + \hat{a} \cdot \hat{b} + 1 + \hat{a} \cdot \hat{b} = 1 \ (\because \hat{a} \cdot \hat{b} = \hat{b} \cdot \hat{a})$$
$$\Rightarrow 2\hat{a} \cdot \hat{b} = -1$$

Now $(\hat{a} - \hat{b})^2 = (\hat{a} - \hat{b}) \cdot (\hat{a} - \hat{b})$

$$= \hat{a} \cdot \hat{a} - \hat{a} \cdot \hat{b} - \hat{b} \cdot \hat{a} + \hat{b} \cdot \hat{b} = 1 - \hat{a} \cdot \hat{b} - \hat{a} \cdot \hat{b} + 1$$
$$= 2 - 2\hat{a} \cdot \hat{b} = 2 - (-1)$$

[Using(i)]

$$= 3$$

$\therefore |\hat{a} - \hat{b}| = \sqrt{3}$

21. If $ABCD$ is a parallelogram and AC and BD are its diagonals, then $\overrightarrow{AC} + \overrightarrow{BD}$ is :

(a) $2\overrightarrow{DA}$

(b) $2\overrightarrow{AB}$

(c) $2\overrightarrow{BC}$

(d) $2\overline{BD}$

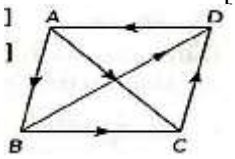
[CBSE 2023]

Solution

(c): Given, $ABCD$ is a parallelogram, then $AB \| CD$ and $BC \| DA$

$\overline{AC} = \overline{AB} + \overline{BC}$ [Triangle law of addition] $\overline{BD} = \overline{BC} + \overline{CD}$ [Triangle law of addition]

$\therefore \overline{AC} + \overline{BD} = \overline{AB} + \overline{BC} + \overline{BC} + \overline{CD} = \overline{AB} + 2\overline{BC} - \overline{AB}$ [$\because \overline{AB} = -\overline{CD}$]



$= 2\overline{BC}$

22. Two vectors $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ are collinear if

(a) $a_1b_1 + a_2b_2 + a_3b_3 = 0$

(b) $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

(c) $a_1 = b_1, a_2 = b_2, a_3 = b_3$

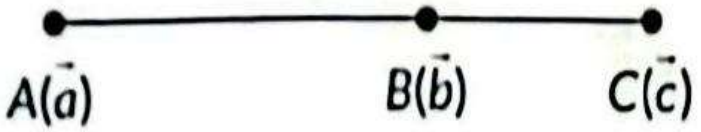
(d) $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$

[CBSE 2023]

Solution

(b)

23. Position vectors of the points A, B and C as shown in the figure below are $\vec{a}, \vec{b}$ and $\vec{c}$ respectively.



If $\overrightarrow{AC} = \frac{5}{4}\overrightarrow{AB}$, express $\vec{c}$ in terms of $\vec{a}$ and $\vec{b}$.

[CBSE 2023]

Solution

Given, position vectors of A, B and C are $\vec{a}, \vec{b}$ and $\vec{c}$ respectively.

and $\overrightarrow{AC} = \frac{5}{4}\overrightarrow{AB}$

$$A(u) \underset{B(\vec{b})}{\overset{C(\vec{c})}{\rightarrow}} \#(i)$$

Now, from the figure $\overrightarrow{AC} = \vec{c} - \vec{a}$,

$\overline{AB} = \bar{b} - \bar{a}$

$\therefore$ By (i), we have $\vec{c} - \vec{a} = \frac{5}{4}(\vec{b} - \vec{a})$

$\Rightarrow \vec{c} = \frac{5}{4}\vec{b} - \frac{5}{4}\vec{a} + \vec{a} \Rightarrow \vec{c} = \frac{5}{4}\vec{b} - \frac{1}{4}\vec{a} \Rightarrow \vec{c} = \frac{1}{4}(5\vec{b} - \vec{a})$

24. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors such that $|\vec{a}| = 7$, $|\vec{b}| = 24, |\vec{c}| = 25$ and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then find the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$.

[CBSE 2023

Solution

. Given, $|\vec{a}| = 7, |\vec{b}| = 24, |\vec{c}| = 25$ and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

Now, $(\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) = \vec{0} \cdot \vec{0}$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0 \Rightarrow 7^2 + 24^2 + 25^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a})$
$= 0 \Rightarrow 49 + 576 + 625 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0 \therefore \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}$
$= -625$

25. If the vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 3, |\vec{b}| = \frac{2}{3}$ and $\vec{a} \times \vec{b}$ is a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$.

[CBSE 2023]

Solution

We know that, $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

$\Rightarrow\ 1 = 3 \times \frac{2}{3}\sin\theta$ ($\because \vec{a} \times \vec{b}$ is aunit vector)

$\Rightarrow \sin\theta = \frac{1}{2} = \sin 30° \Rightarrow \theta = 30°$

So, angle between $\vec{a}$ and $\vec{b}$ is 30°.

26. If $\vec{a}$ and $\vec{b}$ are two non-collinear vectors, then find x, such that $\vec{\alpha} = (x-2)\vec{a} + \vec{b}$ and $\vec{\beta} = (3 + 2x)\vec{a} - 2\vec{b}$ are collinear.

[CBSE 2025]

Solution

Given vectors $\vec{\alpha}$ and $\vec{\beta}$ are collinear.

$\therefore \beta = k\vec{\alpha} \Rightarrow\ (3+2x)\vec{a} - 2\vec{b} = k\cdot((x-2)\vec{a} + \vec{b}) \Rightarrow (3 + 2x - kx + 2k)\vec{a} + (-2-k)\vec{b} = 0$

Since, $\vec{a}$ and $\vec{b}$ are non collinear vectors.

$\therefore\ -2 - k = 0 \Rightarrow k = -2$

and $3 + 2x - kx + 2k = 0$

$\Rightarrow 3 + 2x + 2x - 4 = 0 \Rightarrow 4x - 1 = 0 \Rightarrow x = \frac{1}{4}$

27. Vector $\vec{r}$ is inclined at equal angles to the three axes x, y and z. If magnitude of $\vec{r}$ is $5\sqrt{3}$ units, then find $\vec{r}$.

[CBSE 2025]

Solution

Let the line makes the angle α, β and γ with the coordinate axes.

Given, $\alpha = \beta = \gamma$

Now, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow 3\cos^2\alpha = 1 \Rightarrow \cos\alpha = \pm\frac{1}{\sqrt{3}} \therefore (l, m, n) = \pm\frac{1}{\sqrt{3}}(1,1,1) \therefore \hat{r} = \left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$

Since, $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$

$\Rightarrow \vec{r} = \hat{r}|\vec{r}| = 5\sqrt{3}\hat{r} = 5\sqrt{3}\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$

$= \pm 5(1,1,1)$ or $\pm 5(\hat{\imath} + \hat{\jmath} + \hat{k})$

Hence, the required vector $\bar{r}$ is $\pm 5(\hat{\imath} + \hat{\jmath} + \hat{k})$.

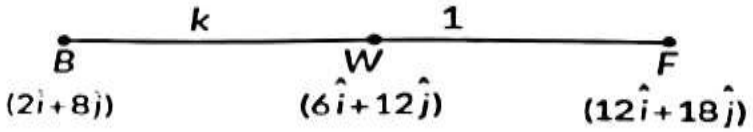
28. During a cricket match, the position of the bowler, the wicket keeper and the leg slip fielder are in a line given by $\vec{B} = 2\hat{\imath} + 8\hat{\jmath}, \vec{W} = 6\hat{\imath} + 12\hat{\jmath}$ and $\vec{F} = 12\hat{\imath} + 18\hat{\jmath}$, respectively. Calculate the ratio in which the wicketkeeper divides the line segment joining the bowler and the leg slip fielder.

[CBSE 2025]

Solution

Given, $\vec{B} = 2\hat{\imath} + 8\hat{\jmath}, \bar{W} = 6\hat{\imath} + 12\hat{\jmath}, \vec{F} = 12\hat{\imath} + 18\hat{\jmath}$

Let wicketkeeper divides the line segment in k: 1 ratio.



Using section formula, we have

$6\hat{\imath} + 12\hat{\jmath} = \frac{k(12\hat{\imath} + 18\hat{\jmath}) + 1(2\hat{\imath} + 8\hat{\jmath})}{k+1} \Rightarrow 6(k+1)\hat{\imath} + 12(k+1)\hat{\jmath} = (12k+2)\hat{\imath} + (18k+8)\hat{\jmath}$

On comparing, we get $6(k+1) = 12k + 2$

$\Rightarrow 6k + 6 = 12k + 2 \Rightarrow 6k = 4 \Rightarrow k = \frac{2}{3}$

Hence, the line segment is divided in the ratio $2:3$.

29. If $\vec{a} + \vec{b} + \vec{c} = \vec{0}, |\vec{a}| = \sqrt{37}, |\vec{b}| = 3$ and $|\vec{c}| = 4$, then angle between $\vec{b}$ and $\vec{c}$ is

(a) $\frac{\pi}{6}$

(b) $\frac{\pi}{4}$

(c) $\frac{\pi}{3}$

(d) $\frac{\pi}{2}$

[CBSE 2025]

Solution

(c): We have, $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$|\vec{a}| = \sqrt{37}, |\vec{b}| = 3$ and $|\vec{c}| = 4$

Now, $\vec{b} + \vec{c} = -\vec{a}$

Squaring both sides, $|\vec{b}|^2 + |\vec{c}|^2 + 2\vec{b} \cdot \vec{c} = |\vec{a}|^2$

$\Rightarrow (3)^2 + (4)^2 + 2(12)\cos\theta = (\sqrt{37})^2 \Rightarrow 9 + 16 + 24\cos\theta = 37$

$\Rightarrow 24\cos\theta = 12 \Rightarrow \cos\theta = \frac{1}{2}$ or $\theta = \frac{\pi}{3}$

$\therefore$ The angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{3}$.

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Three-Dimensional Geometry

1. If a line makes angles 90°, 135°, 45° with the x, y and z axes respectively, find its direction cosines.

[CBSE 2019]

Solution

Since the line makes angles $90°, 135°$ and $45°$ with x, y and z axes respectively.

$$\therefore\ l = \cos 90° = 0, m = \cos 135° = -\frac{1}{\sqrt{2}}$$

and $n = \cos 45° = \frac{1}{\sqrt{2}}$

Hence, direction cosines of the line are

$0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$.

2. Find all the possible vectors of magnitude $5\sqrt{3}$ which are equally inclined to the coordinate axes.

[CBSE 2021-22]

Solution

$\pm 5\sqrt{3}\left(\frac{1}{\sqrt{3}}\hat{\imath} + \frac{1}{\sqrt{3}}\hat{\jmath} + \frac{1}{\sqrt{3}}\hat{k}\right)$ are two possible vectors of magnitude $5\sqrt{3}$ which are equally inclined to the coordinate axes.

3. The Cartesian equation of a line AB is:

$$\frac{2x-1}{12} = \frac{y+2}{2} = \frac{z-3}{3}$$

Find the direction cosines of a line parallel to line AB.

[**CBSE 2021-22]**

Solution

The cartesian equation of line AB is

$\frac{2x-1}{12} = \frac{y+2}{2} = \frac{z-3}{3}$

can be rearranged as $\frac{x-\frac{1}{2}}{6} = \frac{y+2}{2} = \frac{z-3}{3}$

So, $a = 6, b = 2, c = 3$

$\Rightarrow \sqrt{a^2+b^2+c^2} = \sqrt{6^2+2^2+3^2} = 7$

$\therefore$ Required direction cosines are $\frac{6}{7}, \frac{2}{7}, \frac{3}{7}$.

4. If the cartesian equation of a line is $\frac{3-x}{5} = \frac{y+4}{7} = \frac{2z-6}{4}$, write the vector equation for the line.

[CBSE 2014]

Solution

The cartesian equation of a line is

$\frac{3-x}{5}=\frac{y+4}{7}=\frac{2z-6}{4}$

$\Rightarrow \frac{x-3}{-5}=\frac{y-(-4)}{7}=\frac{z-3}{2}=\lambda$ (say)

$\Rightarrow x=3-5\lambda, y=-4+7\lambda, z=3+2\lambda$

Take $\vec{a}=3\hat{\imath}-4\hat{\jmath}+3\hat{k}$ and $\vec{b}=-5\hat{\imath}+7\hat{\jmath}+2\hat{k}$.

$\therefore$ The vector equation of the line (i) is $\vec{r}=\vec{a}+\lambda\vec{b}$

$\Rightarrow \vec{r}=(3\hat{\imath}-4\hat{\jmath}+3\hat{k})+\lambda(-5\hat{\imath}+7\hat{\jmath}+2\hat{k})$

5. Write the vector equation of a line passing through the point $(1,-1,2)$ and parallel to the line whose equation is $\frac{x-3}{1}=\frac{y-1}{2}=\frac{z+1}{-2}$.

[CBSE 2013C]

Solution

Vector eq. of the line passing through $(1,-1,2)$ and parallel to the line, $\frac{x-3}{1}=\frac{y-1}{2}=\frac{z+1}{-2}$ is

$$\vec{r}=(\hat{\imath}-\hat{\jmath}+2\hat{k})+\lambda(\hat{\imath}+2\hat{\jmath}-2\hat{k})$$

6. Find the cartesian equation of the line which passes through the point $(-2,4,-5)$ and is parallel to the line $\frac{x+3}{3}=\frac{4-y}{5}=\frac{z+8}{6}$.

[CBSE 2013]

Solution

. Equation of the line can be written as

$\frac{x+3}{3}=\frac{y-4}{-5}=\frac{z+8}{6}$

Direction ratios of this line are $3,-5,6$.

The required line passes through $(-2,4,-5)$ and its direction ratios are proportional to $3,-5,6$. So, its equation is $\frac{x+2}{3}=\frac{y-4}{-5}=\frac{z+5}{6}$

7. The vector equation of a line which passes through the points $(3,4,-7)$ and $(1,-1,6)$ is_____.

[CBSE 2020]

Solution

Vector equation of a line passes through the points $(3,4,-7)$ and $(1,-1,6)$ is given by

$$\vec{r}=(3\hat{\imath}+4\hat{\jmath}-7\hat{k})+\lambda[(\hat{\imath}-\hat{\jmath}+6\hat{k})-(3\hat{\imath}+4\hat{\jmath}-7\hat{k})]$$

$$\therefore\ \vec{r}=3\hat{\imath}+4\hat{\jmath}-7\hat{k}+\lambda(-2\hat{\imath}-5\hat{\jmath}+13\hat{k})$$

Based on Angle between Two Lines

8. Find the angle between the lines

$\vec{r} = 2\hat{\imath} - 5\hat{\jmath} + \hat{k} + \lambda(3\hat{\imath} + 2\hat{\jmath} + 6\hat{k})$ and

$\vec{r} = 7\hat{\imath} - 6\hat{k} + \mu(\hat{\imath} + 2\hat{\jmath}) + 2\hat{k})$.

[CBSE 2014]

Solution

For $\vec{r} = 2\hat{\imath} - 5\hat{\jmath} + \hat{k} + \lambda(3\hat{\imath} + 2\hat{\jmath} + 6\hat{k})$,

$$\vec{b}_1 = 3\hat{\imath} + 2\hat{\jmath} + 6\hat{k}$$

Also for

$$\vec{r} = 7\hat{\imath} - 6\hat{k} + \mu(\hat{\imath} + 2\hat{\jmath} + 2\hat{k}), \vec{b}_2 = \hat{\imath} + 2\hat{\jmath} + 2\hat{k}$$

Let θ be the angle between the lines.

So, $\theta = \cos^{-1}\left|\frac{\vec{b}_1 \cdot \vec{b}_2}{||\vec{b}_1||\vec{b}_2|}\right|$

$$\Rightarrow \theta = \cos^{-1}\left|\frac{(3\hat{\imath} + 2\hat{\jmath} + 6\hat{k}) \cdot (\hat{\imath} + 2\hat{\jmath} + 2\hat{k})}{\sqrt{3^2 + 2^2 + 4^2}\sqrt{1^2 + 2^2 + 2^2}}\right|$$

$$\Rightarrow \theta = \cos^{-1}\left|\frac{3 + 4 + 12}{7 \times 3}\right| \Rightarrow \theta = \cos^{-1}\left(\frac{19}{21}\right)$$

9. Find the value of k so that the lines $x = -y = kz$ and $x - 2 = 2y + 1 = -z + 1$ are perpendicular to each other.

[CBSE 2020]

Solution

The given lines are

$$I_1: \frac{x - 0}{1} = \frac{y - 0}{-1} = \frac{z - 0}{\frac{1}{k}}$$

$$I_2: \frac{x - 2}{1} = \frac{y + \frac{1}{2}}{\frac{1}{2}} = \frac{z - 1}{-1}$$

$\because$ I_1 is perpendicular to I_2

$$\therefore 1(1) + (-1)\left(\frac{1}{2}\right) + \left(\frac{1}{k}\right)(-1) = 0$$

$$\Rightarrow 1 - \frac{1}{2} - \frac{1}{k} = 0 \Rightarrow \frac{1}{2} = \frac{1}{k} \Rightarrow k = 2$$

10. Find the shortest distance between the lines

$\vec{r} = (4\hat{\imath} - \hat{\jmath}) + \lambda(\hat{\imath} + 2\hat{\jmath} - 3\hat{k})$ and $\vec{r} = (\hat{\imath} - \hat{\jmath} + 2\hat{k}) + \mu(2\hat{\imath} + 4\hat{\jmath} - 5\hat{k})$.

[CBSE 2018]

Solution

We have, $\vec{r} = (4\hat{\imath} - \hat{\jmath}) + \lambda(\hat{\imath} + 2\hat{\jmath} - 3\hat{k})$
$\vec{r} = (\hat{\imath} - \hat{\jmath} + 2\hat{k}) + \mu(2\hat{\imath} + 4\hat{\jmath} - 5\hat{k})$
Comparing with lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$, we get

$$\vec{a}_1 = (4\hat{\imath} - \hat{\jmath}),\ \vec{b}_1 = \hat{\imath} + 2\hat{\jmath} - 3\hat{k}$$
$$\vec{a}_2 = \hat{\imath} - \hat{\jmath} + 2\hat{k},\ \vec{b}_2 = 2\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$$
$$\text{Now, } \vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix}$$
$$= \hat{\imath}(-10 + 12) - \hat{\jmath}(-5 + 6) + \hat{k}(4 - 4)$$
$$= 2\hat{\imath} - \hat{\jmath}$$
$$\text{and } |\vec{b}_1 \times \vec{b}_2| = \sqrt{(2)^2 + (1)^2} = \sqrt{5}$$
$$\vec{a}_2 - \vec{a}_1 = -3\hat{\imath} + 2\hat{k}$$

$\therefore$ Shortest distance, $d = \left|\frac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right|$

$$= \left|\frac{(2\hat{\imath} - \hat{\jmath}) \cdot (-3\hat{\imath} + 2\hat{k})}{\sqrt{5}}\right| = \left|\frac{-6 + 0 + 0}{\sqrt{5}}\right| = \frac{6}{\sqrt{5}}$$

11. The x-coordinate of a point on the line joining the points P(2, 2, 1) and Q(5, 1, – 2) is 4. Find its z-coordinate.

[CBSE 2017]

Solution

Given that $P(2,2,1)$ and $Q(5,1,-2)$
Let the point R on the line PQ, divides the line in the ratio $k:1$. And x-coordinate of point R on the line is 4 .
So, by section formula

$$4 = \frac{5k + 2}{k + 1} \Rightarrow k = 2$$

Now, z-coordinate of point R,

$$z = \frac{-2k + 1}{k + 1} = \frac{-2 \times 2 + 1}{2 + 1} = -1$$

$\Rightarrow z$-coordinate of point $R = -1$

12. Find the image of the point $(2, -1, 5)$ in the line $\frac{x-11}{10} = \frac{y+2}{-4} = \frac{z+8}{-11}$. Also find the equation of the line joining the given point and its image. Find the length of that line segment also.

[CBSE 2013C]

Solution

The given line is

$$\frac{x - 11}{10} = \frac{y + 2}{-4} = \frac{z + 8}{-11} = \lambda \text{ (say)}$$

Any point on it is

$$M(10\lambda + 11, -4\lambda - 2, -11\lambda - 8)$$

Let the given point be $P(2,-1,5)$

D.r's of PM are $10\lambda+11-2,-4\lambda-2+1,-11\lambda-8-5$.

i.e., $10\lambda+9,-4\lambda-1,-11\lambda-13$.

As line (i) is $\perp$ to PM, so

$$(10\lambda+9)\cdot 10+(-4\lambda-1)\cdot(-4)+(-11\lambda-13)\cdot(-11)=0$$
$$\Rightarrow (100+16+121)\lambda+(90+4+143)=0$$
$$\Rightarrow 237\lambda+237=0\Rightarrow\lambda=-1.$$

$\therefore$ D.r's of PM are $-1,3,-2$ i.e., $1,-3,2$.

$\therefore$ Eq. of the line PM is

$$\frac{x-2}{1}=\frac{y+1}{-3}=\frac{z-5}{2}$$

If $Q(\alpha,\beta,\gamma)$ is the image of P in line (i), then M is the mid point of PQ.

Here, $M\equiv(1,2,3)$

$$\therefore \frac{\alpha+2}{2}=1,\frac{\beta-1}{2}=2,\frac{\gamma+5}{2}=3$$
$$\Rightarrow \alpha=0,\beta=5,\gamma=1$$

$\therefore$ $Q(0,5,1)$ is the image of P in line (i)

and $$PQ=\sqrt{(2-0)^2+(-1-5)^2+(5-1)^2}$$
$$=\sqrt{56}=2\sqrt{14}\text{ units}$$

13. Find the vector and cartesian equations of the line through the point $(1,2,-4)$ and perpendicular to the two lines

$\vec{r}=\left(8\hat{\imath}-19\hat{\jmath}+10\hat{k}\right)+\lambda\left(3\hat{\imath}-16\hat{\jmath}+7\hat{k}\right)$ and $\vec{r}=\left(15\hat{\imath}+29\hat{\jmath}+5\hat{k}\right)+\mu\left(3\hat{\imath}+8\hat{\jmath}-5\hat{k}\right)$.

[CBSE 2016, 2015]

Solution

The given lines are

$$\vec{r}=(8\hat{\imath}-19\hat{\jmath}+10\hat{k})+\lambda(3\hat{\imath}-16\hat{\jmath}+7\hat{k})$$
and $$\vec{r}=(15\hat{\imath}+29\hat{\jmath}+5\hat{k})+\mu(3\hat{\imath}+8\hat{\jmath}-5\hat{k})$$

Equation of any line through $(1,2,-4)$ with d.r's I,m,n is

$$\vec{r}=(\hat{\imath}+2\hat{\jmath}-4\hat{k})+p(l\hat{\imath}+m\hat{\jmath}+n\hat{k})$$

Since, the required line is perpendicular to both the given lines.

$$\therefore\ 31-16m+7n=0 \text{ and } 31+8m-5n=0$$
$$\Rightarrow \frac{1}{80-56}=\frac{m}{21+15}=\frac{n}{24+48}\Rightarrow\frac{1}{2}=\frac{m}{3}=\frac{n}{6}$$

$\therefore$ From (i), the required line is

$$\vec{r}=(\hat{\imath}+2\hat{\jmath}-4\hat{k})+p(2\hat{\imath}+3\hat{\jmath}+6\hat{k})$$

Here, the position vector of passing point is $\vec{a}=\hat{\imath}+2\hat{\jmath}-4\hat{k}$ and parallel vector is $\vec{b}=2\hat{\imath}+3\hat{\jmath}+6\hat{k}$

$\therefore$ Cartesian equation of line is given by

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+4}{6}$$

14. Find the value of p, so that the lines $I_1 : \frac{1-x}{3} = \frac{7y-14}{p} = \frac{z-3}{2}$ and $I_2 : \frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$ are perpendicular to each other. Also find the equation of a line passing through a point $(3, 2, -4)$ and parallel to line I_1.

[CBSE 2014]

Solution

The given lines are

$l_1: \frac{x-1}{-3} = \frac{y-2}{p/7} = \frac{z-3}{2}$

$I_2: \frac{x-1}{-3p/7} = \frac{y-5}{1} = \frac{z-6}{-5}$

$\because$ I_1 is perpendicular to I_2.

$\therefore\ (-3)\left(\frac{-3p}{7}\right) + \frac{p}{7} \cdot 1 + 2(-5) = 0$

$\Rightarrow \frac{9p}{7} + \frac{p}{7} = 10 \Rightarrow \frac{10p}{7} = 10 \Rightarrow p = 7$

Now, eq. of the line passing through $(3,2,-4)$ and parallel to I_1 is

$$\frac{x-3}{-3} = \frac{y-2}{1} = \frac{z+4}{2}$$

15. Distance of the point (p, q, r) from y-axis is

(a) q

(b) $|q|$

(c) $|q| + |r|$

(d) $\sqrt{p^2 + r^2}$

[CBSE 2023]

Solution

(d): Given point is (p, q, r)

The foot of perpendicular drawn from point (p, q, r) on the y-axis is $(0, q, 0)$.

Now, distance between these two points is

$$\sqrt{(p-0)^2 + (q-q)^2 + (r-0)^2} = \sqrt{p^2 + r^2}$$

16. The vector equation of a line which passes through the point $(2, -4, 5)$ and is parallel to the line $\frac{x+3}{3} = \frac{4-y}{2} = \frac{z+8}{6}$ is :

(a) $\bar{r} = (-2\hat{\imath} + 4\hat{\jmath} - 5\hat{k}) + \lambda(3\hat{\imath} + 2\hat{\jmath} + 6\hat{k})$

(b) $\vec{r} = (2\hat{\imath} - 4\hat{\jmath} + 5\hat{k}) + \lambda(3\hat{\imath} - 2\hat{\jmath} + 6\hat{k})$

(c) $\vec{r} = (2\hat{\imath} - 4\hat{\jmath} + 5\hat{k}) + \lambda(3\hat{\imath} + 2\hat{\jmath} + 6\hat{k})$

(d) $\vec{r} = (-2\hat{\imath} + 4\hat{\jmath} - 5\hat{k}) + \lambda(3\hat{\imath} - 2\hat{\jmath} - 6\hat{k})$

[CBSE 2023]

Solution

(b): Let $\bar{a}$ be the position vector of the point $(2, -4, 5)$, then $\bar{a} = 2\hat{\imath} - 4\hat{\jmath} + 5\hat{k}$.

The given equation of line is $\frac{x+3}{3} = \frac{4-y}{2} = \frac{z+8}{6}$

$$\Rightarrow \frac{x+3}{3} = \frac{y-4}{-2} = \frac{z+8}{6} \#(i)$$

$\therefore$ Direction ratios of (i) are $3, -2, 6$.

Let $\vec{b}$ be the vector parallel to line (i).

Then, $\vec{b} = 3\hat{\imath} - 2\hat{\jmath} + 6\hat{k}$

$\therefore$ The vector equation of required line is $\bar{r} = \bar{a} + \lambda\bar{b}$

i.e. $\vec{r} = (2\hat{\imath} - 4\hat{\jmath} + 5\hat{k}) + \lambda(3\hat{\imath} - 2\hat{\jmath} + 6\hat{k})$

17. Show that the lines $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}$ intersect. Also find their point of intersection.

[CBSE 2023]

Solution

Any point on the line

$\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7} = r$ (say)

is of the form $(3r - 1, 5r - 3, 7r - 5)$.

Any point on the line

$\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5} = k$ (say)

is of the form $(k + 2, 3k + 4, 5k + 6)$

For lines (i) and (ii) to intersect, we must have

$3r - 1 = k + 2, 5r - 3 = 3k + 4, 7r - 5 = 5k + 6$

On solving these, we get $r = \frac{1}{2}, k = -\frac{3}{2}$

$\therefore$ Point of intersection is $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$

18. The angle between the lines $2x = 3y = -z$ and $6x = -y = -4z$ is

(a) $0°$

(b) $30°$

(c) $45°$

(d) $90°$

[CBSE 2023]

Solution

(d):The given equation of lines can be rewritten as

$\frac{x-0}{1/2} = \frac{y-0}{1/3} = \frac{z-0}{-1}$ and $\frac{x-0}{1/6} = \frac{y-0}{-1} = \frac{z-0}{-1/4}$

$\therefore a_1 = \frac{1}{2}, b_1 = \frac{1}{3}, c_1 = -1$ and $a_2 = \frac{1}{6}, b_2 = -1, c_2 = \frac{-1}{4}$

Now, $\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$

$$= \frac{\frac{1}{2}\cdot\frac{1}{6} + \frac{1}{3}\cdot(-1) + (-1)\cdot\left(\frac{-1}{4}\right)}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2 + (-1)^2}\sqrt{\left(\frac{1}{6}\right)^2 + (-1)^2 + \left(\frac{-1}{4}\right)^2}} = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = 90°$$

19. Find the coordinates of the foot of the perpendicular drawn from point $(5,7,3)$ to the line $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$.

[CBSE 2023]

Solution

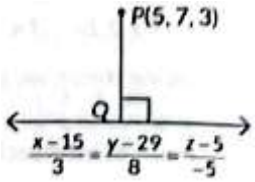
We have point $P(5,7,3)$ and equation of line as

$$\frac{x - 15}{3} = \frac{y - 29}{8} = \frac{z - 5}{-5} = k(\text{ say})$$

Any point on this line is given by

$$Q(3k+15, 8k+29, -5k+5)$$

Direction ratio of line PQ are



$$\langle 3k+15-5, 8k+29-7, -5k+5-3\rangle$$

i.e., $\langle 3k+10, 8k+22, -5k+2\rangle$

As, PQ is perpendicular to given line

$\therefore\ 3(3k+10)+8(8k+22)-5(-5k+2)=0 \Rightarrow 98k+196=0 \Rightarrow\ k=-2$

$\therefore$ Foot of perpendicular drawn from given point

$P(5,7,3)$ on the given line is

$(-6+15, -16+29, 10+5)$ i.e., $(9,13,15)$

20. Find the shortest distance between the pair of lines $\frac{x-1}{2}=\frac{y+1}{3}=z$ and $\frac{x+1}{5}=\frac{y-2}{1}; z=2$.

[CBSE 2023]

Solution

Given, lines are

$$\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-0}{1} \text{ and } \frac{x+1}{5}=\frac{y-2}{1}=\frac{z-2}{0}$$

Here, $x_1=1, y_1=-1, z_1=0; x_2=-1, y_2=2, z_2=2$

$a_1=2, b_1=3, c_1=1; a_2=5, b_2=1, c_2=0$

Let d be the required shortest distance, then

$$d=\left|\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}\right|$$

$$=\left|\frac{\begin{vmatrix} -2 & 3 & 2 \\ 2 & 3 & 1 \\ 5 & 1 & 0 \end{vmatrix}}{\sqrt{(0-1)^2+(5-0)^2+(2-15)^2}}\right| = \left|\frac{-2(0-1)-3(0-5)+2(2-15)}{\sqrt{1+25+169}}\right|$$

$$=\left|\frac{2+15-26}{\sqrt{195}}\right|=\frac{9}{\sqrt{195}} \text{ units}$$

21. Find the distance between the line $\frac{x}{2}=\frac{2y-6}{4}=\frac{1-z}{-1}$ and another line parallel to it passing through the point $(4,0,-5)$.

[CBSE 2024]

Solution

Equation of line passing through the point $(4,0,-5)$ and parallel to the line $\frac{x}{2}=\frac{2y-6}{4}=\frac{1-z}{-1}$ is

$$\frac{x-4}{2}=\frac{y-0}{2}=\frac{z+5}{1}$$

Here, $\bar{a}_1=0\hat{\imath}+3\hat{\jmath}+\hat{k}, \vec{a}_2=4\hat{\imath}+0\hat{\jmath}-5\hat{k}, \vec{b}=2\hat{\imath}+2\hat{\jmath}+\hat{k}$,

$\bar{a}_2-\bar{a}_1=4\hat{\imath}-3\hat{\jmath}-6\hat{k}$

Distance between parallel lines is $d = \left|\frac{\vec{b}\times(\bar{a}_2-\bar{a}_1)}{|\vec{b}|}\right|$.

Now, $\bar{b}\times(\bar{a}_2-\bar{a}_1) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 2 & 1 \\ 4 & -3 & -6 \end{vmatrix}$

$= \hat{\imath}(-12+3) - \hat{\jmath}(-12-4) + \hat{k}(-6-8) = -9\hat{\imath} + 16\hat{\jmath} - 14\hat{k}$

$d = \left|\frac{\sqrt{81+256+196}}{\sqrt{4+4+1}}\right| = \frac{\sqrt{533}}{3}$ units

22. The line $x = 1+5\mu, y = -5+\mu, z = -6-3\mu$ passes through which of the following point ?

(a) $(1,-5,6)$

(b) $(1,5,6)$

(c) $(1,-5,-6)$

(d) $(-1,-5,6)$

[CBSE 2025]

Solution

(c) : Given, $x = 1+5\mu, y = -5+\mu, z = -6-3\mu$

$$\Rightarrow \frac{x-1}{5} = \frac{y+5}{1} = \frac{z+6}{-3} = \mu\#(i)$$

From given options, line (I) passes through point (1, -5,-6)

$\therefore$ We have, $\mu = \frac{1-1}{5} = \frac{-5+5}{1} = \frac{-6+6}{-3} = 0$

$\therefore$ Point $(1,-5,-6)$ satisfies the equation of given line.

23. Find the distance of the point $P(2,4,-1)$ from the line $\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}$.

[CBSE 2025]

Solution

Given, equation of line $\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}$

Let L be any point of the given line, then

$\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9} = \lambda$ (Say)

$\Rightarrow x = \lambda-5, y = 4\lambda-3, z = -9\lambda+6$

Let the coordinate of L be $(\lambda-5, 4\lambda-3, -9\lambda+6)$,

Dr's of PL are $(\lambda-7, 4\lambda-7, 7-9\lambda)$

Also, direction ratio of given line are proportional to $1,4,-9$.

Since, PL is perpendicular to the given line

$\therefore\ 1(\lambda-7)+4(4\lambda-7)-9(7-9\lambda) = 0 \Rightarrow \lambda-7+16\lambda-28-63+81\lambda = 0 \Rightarrow 98\lambda = 98 \Rightarrow \lambda = 1$

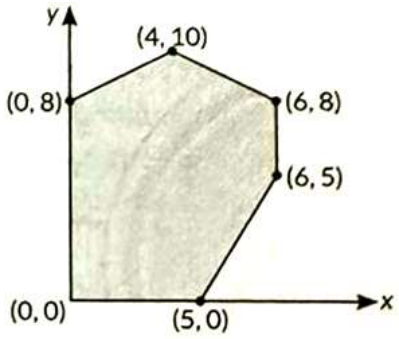
So, the coordinates of L are $(-4,1,-3)$

$\therefore$ Required distance $= \sqrt{(-4-2)^2+(1-4)^2+(-3+1)^2}$

$= \sqrt{36+9+4} = \sqrt{49} = 7$ units

LPP

1. The feasible region for an LPP is shown below: Let $z = 3x - 4y$ be the objective function. Minimum of z occurs at



(a) (0,0)
(b)(0,8)
(c) (5,0)
(d) (4,10)
(CBSE 2020)

Solutions

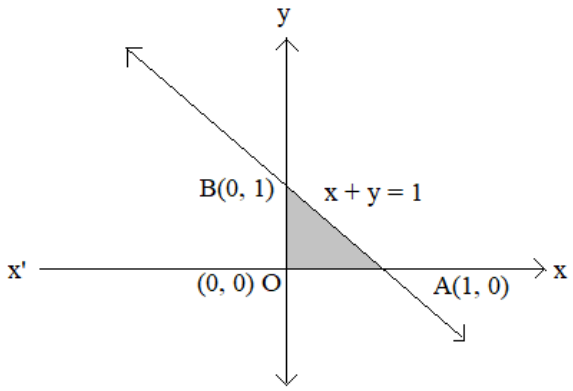
(b) : We know that minimum of objective function occurs at corner points.

Corner points	Value of $z = 3x - 4y$
(0,0)	0
(5,0)	15
(6,5)	-2
(6,8)	-14
(4,10)	-28
(0,8)	$-32 \leftarrow$ Minimum

2. The maximum value of $Z = 3x + 4y$ subject to the constraints $x \geq 0, y \geq 0$ and $x + y \leq 1$ is
(a) 7
(b) 4
(c) 3
(d) 10
(CBSE 2021-22)

Solutions

(b): We have to maximise $Z = 3x + 4y$
Subject to constraints, $x \geq 0, y \geq 0$ and $x + y \leq 1$



The shaded portion OAB is the feasible region, where $O(0,0), A(1,0)$ and $B(0,1)$ are the corner points.

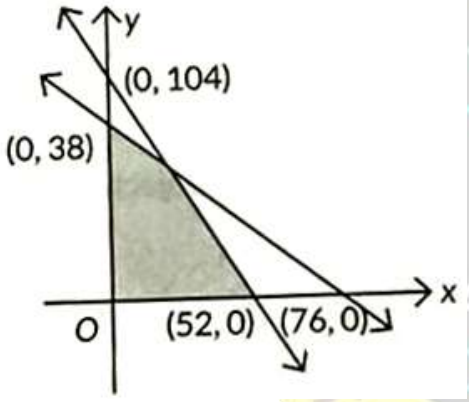
At $O(0,0), Z = 3 \times 0 + 4 \times 0 = 0$

At $A(1,0), Z = 3 \times 1 + 4 \times 0 = 3$

At $B(0,1), Z = 3 \times 0 + 4 \times 1 = 4$

$\therefore$ Maximum value of Z is 4 , which occurs at $B(0,1)$.

3. The feasible region of an LPP is given in the following figure



Then, the constraints of the LPP are $x \geq 0, y \geq 0$ and

(a) $2x + y \leq 52$ and $x + 2y \leq 76$

(b) $2x + y \leq 104$ and $x + 2y \leq 76$

(c) $x + 2y \leq 104$ and $2x + y \leq 76$

(d) $x + 2y \leq 104$ and $2x + y \leq 38$

(CBSE 2021-22)

Solutions

(b): Clearly, the pair of points given in graph, and $(0,104)$; $(52,0)$ and $(0,38)$; $(76,0)$ satisfy the corresponding equations given in option(b) i.e., $2x + y \leq 104$ and $x + 2y \leq 76$

4. The corner points of the feasible region determined by the system of linear inequalities are $(0,0), (4,0), (2,4)$ and $(0,5)$. If the maximum value of $z = ax + by$, where $a, b > 0$ occurs at both $(2,4)$ and $(4,0)$, then

(a) $a = 2b$

(b) $2a = b$

(c) $a = b$

(d) $3a = b$

(CBSE 2020)

Solutions

(a): Since, maximum value of $z = ax +$ by occurs at both (2,4) and (4,0).

$\therefore 2a + 4b = 4a + 0$

$\Rightarrow 4b = 2a \Rightarrow 2b = a$

5. Solve the following LPP graphically:

Minimise $Z = 5x + 10y$

Subject to constraints $x + 2y \leq 120, x + y \geq 60,$

$x - 2y \geq 0$ and $x, y \geq 0$

(CBSE 2017)

Solutions

We have, Minimise $Z = 5x + 10y$,

Subject to constraints :

$x + 2y \leq 120$

$x + y \geq 60$

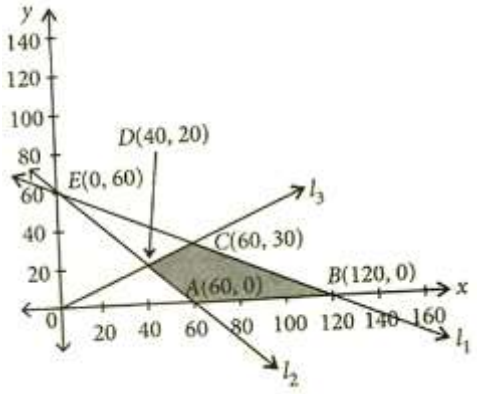
$x - 2y \geq 0$

and $x, y \geq 0$

To solve L.P.P graphically, we convert inequations into equations.

$I_1: x + 2y = 120, I_2: x + y = 60, I_3: x - 2y = 0$ and $x = 0, y = 0$ I_1 and I_2 intersect at $E(0,60)$, I_1 and I_3 intersect at $C(60,30)$, I_2 and I_3 intersect at $D(40,20)$.

The shaded region $ABCD$ is the feasible region and is bounded. The corner points of the feasible region are $A(60,0), B(120,0), C(60,30)$ and $D(40,20)$.



Corner points	Value of $Z = 5x + 10y$
$A(60,0)$	300 ← (Minimum)
$B(120,0)$	600
$C(60,30)$	600
$D(40,20)$	400

Hence, Z is minimum at $A(60,0)$ i.e., 300 .

6. Maximise $Z = x + 2y$
Subject to the constraints:
$x + 2y \geq 100, 2x - y < 0$
$2x + y \leq 200$
$x, y \geq 0$
Solve the above LPP graphically.
(CBSE 2017)

Solutions

Maximise $Z = x + 2y$,
Subject to constraints: $x + 2y \geq 100, 2x - y < 0$, $2x + y \leq 200$ and $x, y \geq 0$.
Converting the inequations into equations, we obtain the lines
$I_1: x + 2y = 100$
$I_2: 2x - y = 0$
$I_3: 2x + y = 200$
$I_4: x = 0$
and $I_5: y = 0$
By intercept form, we get
$I_1: \frac{x}{100} + \frac{y}{50} = 1$
$\Rightarrow$ The line I_1 meets the coordinate axes at $(100,0)$ and $(0,50)$.
$I_2: 2x = y$
$\Rightarrow$ The line I_2 passes through origin and $(50,100)$.
$I_3: \frac{x}{100} + \frac{y}{200} = 1$
$\Rightarrow$ The line I_3 meets the coordinates axes at $(100,0)$ and $(0,200)$.
$I_4: x = 0$ is the y-axis, $I_5: y = 0$ is the x-axis

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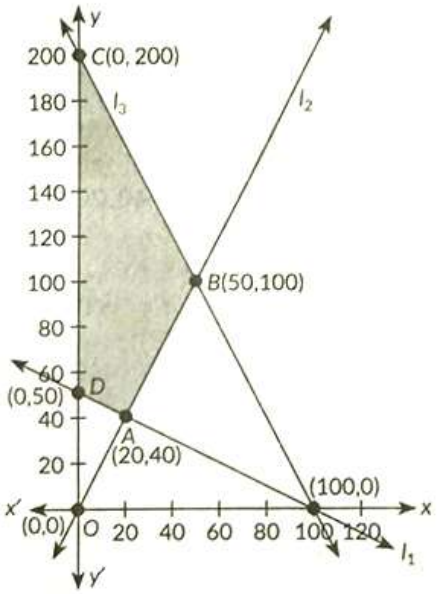


Now, plotting the above points on the graph, we get the feasible region of the LPP as shaded region $ABCD$. The coordinates of the corner points of the feasible region $ABCD$ are $A(20,40), B(50,100), C(0,200), D(0,50)$.

Now, $Z_A = 20 + 2 \times 40 = 100$

$Z_B = 50 + 2 \times 100 = 250, Z_C = 0 + 2 \times 200 = 400$

$Z_D = 0 + 2 \times 50 = 100$

$\therefore$ Z is maximum at $C(0,200)$ and having value 400 .

7. The graph of the inequality $2x + 3y > 6$ is

(a) half plane that contains the origin

(b) half plane that neither contains the origin nor the points of the line $2x + 3y = 6$.

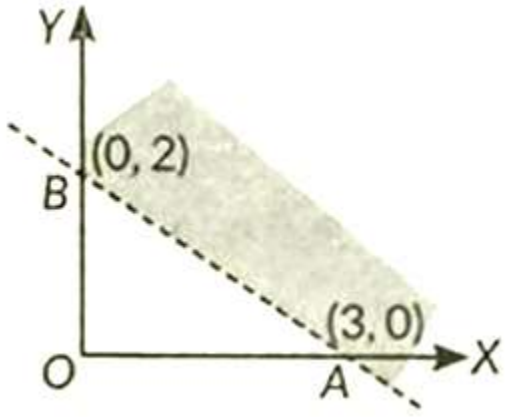
(c) whole XOY-plane excluding the points on the line $2x + 3y = 6$

(d) entire XOY-plane.

(CBSE 2020)

Solutions

(b) : From the graph of inequality $2x + 3y > 6$. It is clear that it does not contain the origin nor the points of the line $2x + 3y = 6$.



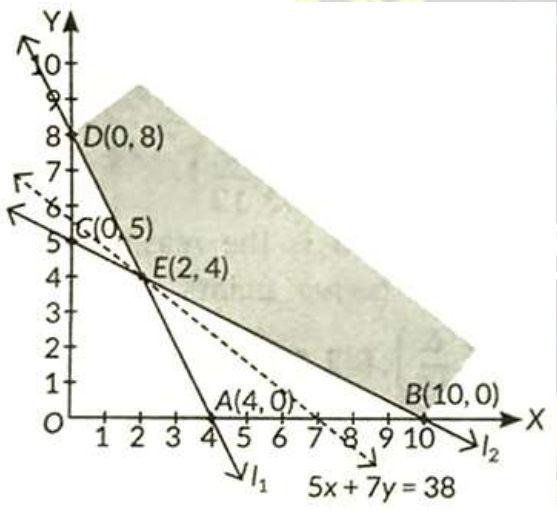
8. Solve the following LPP graphically:
Minimize $z = 5x + 7y$
Subject to the constraints

$$2x + y \geq 8$$
$$x + 2y \geq 10$$
$$x, y \geq 0$$

(CBSE 2020)

Solutions

We have, minimize $z = 5x + 7y$,
Subject to constraints, $2x + y \geq 8, x + 2y \geq 10, x, y \geq 0$
To solve LPP graphically, we convert inequations into equations.
Now, $I_1: 2x + y = 8, I_2: x + 2y = 10$ and $x = 0, y = 0$
I_1 and I_2 intersect at $E(2,4)$
Let us draw the graph of these equations as shown below.



The corner points of the feasible region are $D(0,8)$, $B(10,0)$ and $E(2,4)$.

Corner points	Value of $z = 5x + 7y$
$D(0,8)$	56
$B(10,0)$	50
$E(2,4)$	38 (Minimum)

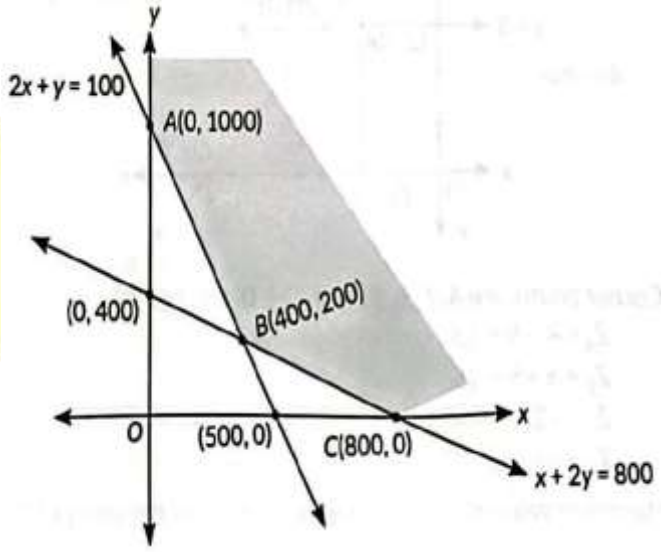
From the table, we find that 38 is the minimum value of z at $E(2,4)$. Since the region is unbounded, so we draw the graph of inequality $5x + 7y < 38$ to check whether the resulting open half plane has any point common with the feasible region. Since it has no point in common. So, the minimum value of z is obtained at $E(2,4)$ and the minimum value of $z = 38$

9. Consider the Linear Programming Problem, where the objective function $Z = (x + 4y)$ need to be minimized subject to constraints
$2x + y \geq 1000x + 2y \geq 800$
$x, y \geq 0$.
Drawaneatgraphofthefeasibleregionand
find the minimum value of Z.

[CBSE 2025]

Solution

. We have, $Z = x + 4y$
Subject to constraints ;
$2x + y \geq 1000; x + 2y \geq 800; x, y \geq 0$
The corner points of feasible region are
$A(0,1000), B(400,200)$ and $C(800,0)$



Corner Points	$Z = x + 4y$
$A(0,1000)$	4000
$B(400,200)$	1200
$C(800,0)$	800

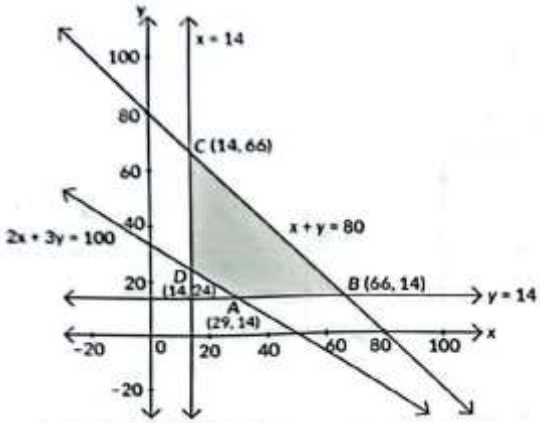
Thus, the minimum value of **Z** is 800 .

10. Solve the following linear programming problem graphically:
Maximise $Z = 20x + 30y$
Subject to the constraints :
$x + y \leq 802x + 3y \geq 100x \geq 14y \geq 14$

[CBSE 2025]

Solution

We have, Max$Z = 20x + 30y$
Subject to constraints, $x + y \leq 80$
$2x + 3y \geq 100 x \geq 14, y \geq 14$



From graph, $ABCD$, the feasible region and the corner points are
$A(29,14), B(66,14), C(14,66), D(14,24)$
The value of Z at these corner points are shown as :

Corner points	$Z = 20x + 30y$
$A(29,14)$	1000
$B(66,14)$	1740
$C(14,66)$	$2260 \rightarrow$ Maximum
$D(14,24)$	1000

Hence, maximum is at $(14,66)$ and maximum value is 2260.

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Probability

1. Out of 8 outstanding students of a school, in which there are 3 boys and 5 girls, a team of 4 students is to be selected for a quiz competition. Find the probability that 2 boys and 2 girls are selected.

[CBSE 2019]

Solution

Total number of students $= 8$

The number of ways to select 4 students out of 8 students

$$= {}^{8}C_4 = \frac{8!}{4!\,4!} = 70$$

The number of ways to select 2 boys and 2 girls

$$= {}^{3}C_2 \times {}^{5}C_2 = \frac{3!}{2!\,1!} \times \frac{5!}{2!\,3!} = 3 \times 10 = 30$$

$\therefore$ Required probability $= \frac{30}{70} = \frac{3}{7}$.

2. A box B_1 contains 1 white ball and 3 red balls. Another box B_2 contains 2 white balls and 3 red balls.

If one ball is drawn at random from each of the boxes B_1 and B_2, then find the probability that the two balls drawn are of the same colour.

[CBSE 2021-22]

Solution

B_1 contains 1 white ball and 3 red balls.

B_2 contains 2 white balls and 3 red balls.

$P($ same colour $) = P($ white ball of B_1 and white ball of $B_2)$ or $P($ red ball of B_1 and red ball of $B_2)$

$$= \frac{1}{4} \times \frac{2}{5} + \frac{3}{4} \times \frac{3}{5}$$

$$= \frac{2}{20} + \frac{9}{20} = \frac{11}{20}$$

3. If A and B are two events such that P(A) = 0.4, P(B) = 0.3 and P(A ∪ B) = 0.6, then find P (B' ∩ A).

[CBSE 2020]

Solution

We have, $P(A) = 0.4, P(B) = 0.3$ and $P(A \cup B) = 0.6$

So, $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$$= 0.4 + 0.3 - 0.6 = 0.1$$

Now, $P(B' \cap A) = P(A - B) = P(A) - P(A \cap B)$

$$= 0.4 - 0.1 = 0.3$$

Based on Conditional Probability

4. 12 cards numbered 1 to 12 (one number on one card), are placed in a box and mixed up thoroughly. Then a card is drawn at random from the box. If it is known that the number on the drawn card is greater than 5, find the probability that the card bears an odd number.

[CBSE 2019]

Solution

The sample space, S is given by

$$S = \{1,2,3,4,5,6,7,8,9,10,11,12\}$$

Let A be the event that number on the drawn card is odd, and B be the event that number on the drawn card is greater than 5 .

$$\therefore\ A = \{1,3,5,7,9,11\}$$
$$B = \{6,7,8,9,10,11,12\}$$

and, $A \cap B = \{7,9,11\}$

Now, $P(A) = \frac{n(A)}{n(S)} = \frac{6}{12}'\ P(B) = \frac{n(B)}{n(S)} = \frac{7}{12}$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{3}{12}$$

Now, $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{3/12}{7/12} = \frac{3}{7}$

Hence, required probability is $\frac{3}{7}$.

5. Assume that each born child is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls? Given that

(i) the youngest is a girl.

(ii) atleast one is a girl.

[CBSE 2014]

Solution

Let $G_i (i = 1,2)$ and $B_i (i = 1,2)$ denote the $i^{t\text{ th}}$ child is a girl or a boy respectively.
Then sample space is,

$$S = \{G_1G_2, G_1B_2, B_1G_2, B_1B_2\}$$

Let A be the event that both children are girls, B be the event that the youngest child is a girl and C be the event that at least one of the children is a girl.

Then $A = \{G_1G_2\}, B = \{G_1G_2, B_1G_2\}$

and $C = \{B_1G_2, G_1G_2, G_1B_2\}$

$\Rightarrow A \cap B = \{G_1G_2\}$ and $A \cap C = \{G_1G_2\}$

(i) Required probability $= P(A/B)$

$$= \frac{P(A \cap B)}{P(B)} = \frac{1/4}{2/4} = \frac{1}{2}$$

(ii) Required probability $= P(A/C)$

$$= \frac{P(A \cap C)}{P(C)} = \frac{1/4}{3/4} = \frac{1}{3}$$

6. Given that E and F are events such that P(E) = 0.8, P(F) = 0.7, P(E ∩ F) = 0.6. Find $P(\bar{E}|\bar{F})$.

[CBSE SQP 2020-21]

Solution

$$P(\bar{E} \mid \bar{F}) = \frac{P(\bar{E} \cap \bar{F})}{P(\bar{F})} = \frac{P(\bar{E} \cup \bar{F})}{P(\bar{F})} = \frac{1 - P(E \cup F)}{1 - P(F)}$$

Now, $P(E \cup F) = P(E) + P(F) - P(E \cap F) = 0.8 + 0.7 - 0.6 = 0.9$

Substituting value of $P(E \cup F)$ in (i), we get

$$P(\bar{E} \mid \bar{F}) = \frac{1 - 0.9}{1 - 0.7} = \frac{0.1}{0.3} = \frac{1}{3}$$

7. Given two independent events A and B such that P(A) = 0.3 and P(B) = 0.6, find P(A'∩B').

[CBSE 2020]

Solution

Given, A and B are independent events. So, A' and B' are also independent events.

Now, $P(A' \cap B') = P(A') \times P(B')$

$= [1 - P(A)][1 - P(B)] = [1 - 0.3][1 - 0.6]$

[Given, $P(A) = 0.3$ and $P(B) = 0.6$]

8. A problem is given to three students whose probabilities of solving it are $\frac{1}{3}, \frac{1}{4}$ and $\frac{1}{6}$ respectively. If the events of solving the problem are independent, find the probability that at least one of them solves it.

[CBSE 2020]

Solution

Let A, B, C be respectively the events of solving problem by three students and $P(A), P(B), P(C)$ be their probability of solving the problem respectively.

$\therefore\ P(A) = \frac{1}{3}, P(B) = \frac{1}{4}$ and $P(C) = \frac{1}{6}$

Required probability $= 1 - P(\bar{A} \cap \bar{B} \cap \bar{C})$

$= 1 - P(\bar{A})P(\bar{B})P(\bar{C})$

($\because A, B, C$ are independent $\therefore \bar{A}, \bar{B}, \bar{C}$ are also independent)

$= 1 - [1 - P(A)][1 - P(B)][1 - P(C)]$

$= 1 - \frac{2}{3} \times \frac{3}{4} \times \frac{5}{6} = 1 - \frac{5}{12} = \frac{7}{12}$

9. A bag A contains 4 black and 6 red balls and bag B contains 7 black and 3 red balls. A die is thrown. If 1 or 2 appears on it, then bag A is chosen, otherwise bag B. If two balls are drawn at random (without replacement) from the selected bag, find the probability of one of them being red and another black.

[CBSE 2015]

Solution

. Probability of choosing bag $A = P(A) = \frac{2}{6} = \frac{1}{3}$

Probability of choosing bag $B = P(B) = \frac{4}{6} = \frac{2}{3}$

Let E_1 and E_2 be the events of drawing a red and a black ball from bag A and B respectively.

$$\therefore P(E_1) = \frac{6 \times 4}{{}^{10}C_2} \text{ and } P(E_2) = \frac{7 \times 3}{{}^{10}C_2}$$

$$\therefore \text{ Required probability } = P(A) \times P(E_1) + P(B) \times P(E_2)$$

$$= \frac{1}{3} \times \frac{6 \times 4}{{}^{10}C_2} + \frac{2}{3} \times \frac{7 \times 3}{{}^{10}C_2} = \frac{8}{45} + \frac{14}{45} = \frac{22}{45}$$

10. There are two bags, I and II. Bag I contains 3 red and 5 black balls and Bag II contains 4 red and 3 black balls. One ball is transferred randomly from Bag I to Bag II and then a ball is drawn randomly from Bag II. If the ball so drawn is found to be black in colour, then find the probability that the transferred ball is also black.

Solution

Let E_1 be the event that ball transferred from bag I to bag I is red and E_2 be the event that ball transferred from bag I to bag II is black. B be the event that ball drawn from bag II is black.

So, $P(E_1) = \frac{3}{8}, P(E_2) = \frac{5}{8}$

$P(B \mid E_1) = \frac{3}{8}, P(B \mid E_2) = \frac{4}{8}$

So, required probability $= P(E_2 \mid B)$

$$= \frac{P(E_2) \times P(B \mid E_2)}{P(E_1) \times P(B \mid E_1) + P(E_2) \times P(B \mid E_2)}$$

$$= \frac{\frac{5}{8} \times \frac{4}{8}}{\frac{3}{8} \times \frac{3}{8} + \frac{5}{8} \times \frac{4}{8}} = \frac{20}{9 + 20} = \frac{20}{29}$$

11. Suppose a girl throws a die. If she gets 1 or 2, she tosses a coin three times and notes the number of tails. If she gets 3, 4, 5 or 6, she tosses a coin once and notes whether a 'head' or 'tail' is obtained. If she obtained exactly one 'tail', what is the probability that she threw 3, 4, 5 or 6 with the die?

[CBSE 2018]

Solution

Let E_1 be the event that the outcome on the die is 1 or 2, E_2 be the event that the outcome on the die is 3,4 , 5,6 .

$\therefore P(E_1) = \frac{2}{6} = \frac{1}{3}$ and $P(E_2) = \frac{4}{6} = \frac{2}{3}$

Let A be the event of getting exactly one tail.

Now, $P(A/E_1)$ be the probability of getting exactly one tail by tossing the coin three times if she gets 1 or $2 = \frac{3}{8}$ and $P(A/E_2)$ be the probability of getting exactly one tail in a single throw of coin if she gets 3,4,5,6 $= \frac{1}{2}$

The probability that the girl threw 3,4,5,6 with the die, if she obtained exactly one tail is given by $P(E_2/A)$.

$$\therefore\ P(E_2/A) = \frac{P(E_2)\cdot P(A/E_2)}{P(E_1)\cdot P(A/E_1) + P(E_2)\cdot P(A/E_2)}$$

$$= \frac{\frac{2}{3}\cdot\frac{1}{2}}{\frac{1}{3}\cdot\frac{3}{8} + \frac{2}{3}\cdot\frac{1}{2}} = \frac{8}{11}$$

12. In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{5}$ be the probability that he knows the answer and $\frac{2}{5}$ be the probability that he guesses. Assuming that a student who guesses the answer will be correct with probability $\frac{1}{3}$, what is the probability that the student knows the answer given that he answered it correctly?

[CBSE 2015C]

Solution

Let E_1, E_2 and A be the events defined as follows:

E_1 : The student knows the answer

E_2 : The student guesses the answer

A : The student answers correctly

We have, $P(E_1) = \frac{3}{5}, P(E_2) = \frac{2}{5}$

Also, $P(A/E_2) = \frac{1}{3}$ and $P(A/E_1) = 1$

$\therefore$ Required probability

$$= P(E_1/A) = \frac{P(E_1)\cdot P(A/E_1)}{P(E_1)\cdot P(A/E_1) + P(E_2)\cdot P(A/E_2)}$$

$$= \frac{\frac{3}{5}\cdot 1}{\frac{3}{5}\cdot 1 + \frac{2}{5}\cdot\frac{1}{3}} = \frac{3\times 3}{3\times 3 + 2} = \frac{9}{11}.$$

13. A card from a pack of 52 playing cards is lost. From the remaining cards of the pack three cards are drawn at random (without replacement) and are found to be all spades. Find the probability of the lost card being a spade.

[CBSE 2014]

Solution

Let E_1, E_2, E_3, E_4 and A be the events defined as below:

E_1 : Missing card is a card of heart.

E_2 : Missing card is a card of spade.

E_3 : Missing card is a card of club.

E_4 : Missing card is a card of diamond.

A : Drawing three spade cards from the remaining cards.

Now, $P(E_1) = P(E_2) = P(E_3) = P(E_4) = \frac{13}{52} = \frac{1}{4}$

$$P(A/E_2) = \frac{{}^{12}C_3}{{}^{11}C_3}$$

$$P(A/E_1) = P(A/E_3) = P(A/E_4) = \frac{{}^{13}C_3}{{}^{51}C_3}$$

$\therefore$ Required probability $= P(E_2/A)$

$$= \frac{P(E_2)P(A/E_2)}{P(A/E_1)P(E_1) + P(A/E_2)P(E_2) + P(A/E_3)P(E_3) + P(A/E_4)P(E_4)}$$

$\therefore$ Required probability $= P(E_2/A)$

$$= \frac{P(E_2)P(A/E_2)}{P(A/E_1)P(E_1) + P(A/E_2)P(E_2) + P(A/E_3)P(E_3) + P(A/E_4)P(E_4)}$$

$$= \frac{\frac{1}{4} \times \frac{{}^{12}C_3}{{}^{51}C_3}}{\frac{1}{4} \times \frac{{}^{13}C_3}{{}^{51}C_3} + \frac{1}{4} \times \frac{{}^{12}C_3}{{}^{51}C_3} + \frac{1}{4} \times \frac{{}^{13}C_3}{{}^{51}C_3} + \frac{1}{4} \times \frac{{}^{13}C_3}{{}^{51}C_3}}$$

$$= \frac{220}{286 + 220 + 286 + 286} = \frac{220}{1078} = \frac{10}{49}$$

14. Assume that the chances of a patient having a heart attack are 50%. Assume that a meditation and yoga course reduces the risk of heart attack by 30% and the prescription of certain drugs and certain restrictions reduces the risk by 25%. At a time a patient chooses only one of the two options with equal probabilities. After going through one of the two options, the patient selected at random suffers a heart attack. Find the probability that the patient followed a course of meditation and yoga. What values are reflected in this question?

[CBSE 2013C]

Solution

Let A, E_1 and E_2 respectively be the events that a person has a heart attack, the selected person followed the course of yoga and meditation and the person adopted the drug prescription.

$$P(A) = \frac{50}{100} = 0.50, P(E_1) = P(E_2) = \frac{1}{2}$$

$$P\left(\frac{A}{E_1}\right) = 0.50 \times 0.70 = 0.35$$

$$P\left(\frac{A}{E_2}\right) = 0.50 \times 0.75 = 0.375$$

Probability that the patient suffering from heart attack followed the course of meditation and yoga is

$$P\left(\frac{E_1}{A}\right) = \frac{P(E_1) \times P\left(\frac{A}{E_1}\right)}{P(E_1) \times P\left(\frac{A}{E_1}\right) + P(E_2) \times P\left(\frac{A}{E_2}\right)}$$

$$= \frac{\frac{1}{2} \times 0.35}{\frac{1}{2} \times 0.35 + \frac{1}{2} \times 0.375} = \frac{0.175}{0.175 + 0.1875} = \frac{14}{29}.$$

Now, $P\left(\frac{E_2}{A}\right) = \frac{P(E_2) \times P\left(\frac{A}{E_2}\right)}{P(E_1) \times P\left(\frac{A}{E_1}\right) + P(E_2) \times P\left(\frac{A}{E_2}\right)}$

$$= \frac{\frac{1}{2} \times 0.375}{\frac{1}{2} \times 0.35 + \frac{1}{2} \times 0.375} = \frac{0.1875}{0.175 + 0.1875} = \frac{15}{29}.$$

Since, $P\left(\frac{E_1}{A}\right) < P\left(\frac{E_2}{A}\right)$. So the course of yoga and meditation is more reflected for a person having heart attack.

15. Five fair coins are tossed simultaneously. The probability of the events that atleast one head comes up is

(a) $\frac{27}{32}$

(b) $\frac{5}{32}$

(c) $\frac{31}{32}$

(d) $\frac{1}{32}$

[CBSE 2023]

Solution

(c) : Since each coin turns up on either a head or tail.

$\therefore$ Total possible outcomes $= 2^5 = 32$

Let A be the event that all tails comes up.

$\therefore$ $\mathbf{n}(A) = 1$ (i.e., (T, T, T, T, T)

So, required probability $= 1 - P(A) = 1 - \frac{1}{32} = \frac{31}{32}$

16. A family has 2 children and the elder child is a girl. The probability that both children are girls is :

(a) $\frac{1}{4}$

(b) $\frac{1}{8}$

(c) $\frac{1}{2}$

(d) $\frac{3}{4}$

[CBSE 2023]

Solution

(c) : Let $G_i (i = 1,2)$ and $B_i (i = 1,2)$ denote the i^{th} child is a girl or a boy respectively.

Then sample space, $S = \{G_1G_2, G_1B_2, B_1G_2, B_1B_2\}$

Let A be the event that both children are girls,

B be the event that the elder child is a girl.

Thus, $A = (G_1G_2)$ and $B = (G_1G_2, G_1B_2)$

$\Rightarrow A \cap B = (G_1G_2)$

$\therefore$ Required probability $= P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{2/4} = \frac{1}{2}$

17. If $P\left(\frac{A}{B}\right) = 0.3, P(A) = 0.4$ and $P(B) = 0.8$, then $P\left(\frac{B}{A}\right)$ is equal to :

(a) 0.6

(b) 0.3

(c) 0.06

(d) 0.4

[CBSE 2023]

Solution

. (a): We have, $P\left(\frac{A}{B}\right) = 0.3, P(A) = 0.4$ and $P(B) = 0.8$

Now, $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$

$\Rightarrow\ 0.3 = \frac{P(A \cap B)}{0.8} \Rightarrow P(A \cap B) = 0.3 \times 0.8 = 0.24$

Now, $P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)} = \frac{0.24}{0.4} = 0.6$

18. For two events A and B, if $P(A) = 0.4, P(B) = 0.8$ and $P(B/A) = 0.6$, then $P(A \cup B)$ is

(a) 0.24

(b) 0.3

(c) 0.48

(d) 0.96

[CBSE 2023]

Solution

(d) : We have, $P(A) = 0.4, P(B) = 0.8$ and $P(B/A) = 0.6$

We know that $P(B/A) = \frac{P(A \cap B)}{P(A)}$

$\Rightarrow 0.6 = \frac{P(A \cap B)}{0.4} \Rightarrow P(A \cap B) = 0.24$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.4 + 0.8 - 0.24 = 0.96$

Hence, $P(A \cup B) = 0.96$

19. Out of two bags, bag A contains 2 white and 3 red balls and bag B contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that it was drawn from bag B.

[CBSE 2023]

Solution

Let E_1, E_2 and A denote the events defined as follows: $E_1 =$ chosen bag is bag A; $E_2 =$ chosen bag is bag B

$A =$ Getting a red ball; $P(E_1) = \frac{1}{2}, P(E_2) = \frac{1}{2}$

$P(A \mid E_1) = \frac{3}{5}$ and $P(A \mid E_2) = \frac{5}{9}$

Using Bayes' theorem

$$P(E_2 \mid A) = \frac{P(E_2) \cdot P(A \mid E_2)}{P(E_1) \cdot P(A \mid E_1) + P(E_2) \cdot P(A \mid E_2)}$$

$$= \frac{\frac{1}{2} \times \frac{5}{9}}{\frac{1}{2} \times \frac{3}{5} + \frac{1}{2} \times \frac{5}{9}} = \frac{\frac{5}{9}}{\frac{3}{5} + \frac{5}{9}} = \frac{\frac{5}{9}}{\frac{52}{45}} = \frac{5}{9} \times \frac{45}{52} = \frac{25}{52}$$

20. If E and F are two independent events such that $P(E) = \frac{2}{3}, P(F) = \frac{3}{7}$, then $P(E/\bar{F})$ is equal to :

(a) $\frac{1}{6}$

(b) $\frac{1}{2}$

(c) $\frac{2}{3}$

(d) $\frac{7}{9}$

[CBSE 2025]

Solution

c): We have, $P(E) = \frac{2}{3}$ and $P(F) = \frac{3}{7}$

$P(E/\bar{F}) = \frac{P(E \cap \bar{F})}{P(\bar{F})} P(\bar{F}) = 1 - P(F) = 1 - \frac{3}{7} = \frac{4}{7}$

So, $P(E/\bar{F}) = \frac{\left(\frac{2}{3}\right) \times \left(\frac{4}{7}\right)}{\frac{4}{7}} = \frac{2}{3}$ [$\because E$ and F areindependent events]

21. Let M and N be two events such that $P(M) = 0.6$, $P(N) = 0.2$ and $P(M \cap N) = 0.5$, then $P(M'/N')$ is

(a) $\frac{7}{8}$

(b) $\frac{2}{5}$

(c) $\frac{1}{2}$

(d) $\frac{2}{3}$

[CBSE 2025]

Solution

(a) : $P(M'/N') = \frac{P(M' \cap N')}{P(N')} = \frac{P((M \cup N)')}{P(N')}$

$$= \frac{1 - P(M \cup N)}{1 - P(N)} = \frac{1 - (P(M) + P(N) - P(M \cap N))}{1 - P(N)}$$

$$= \frac{1 - (0.6 + 0.2 - 0.5)}{1 - 0.2} = \frac{1 - 0.3}{0.8} = \frac{0.7}{0.8} = \frac{7}{8}$$

22. The box has 4 green, 8 blue and 3 red pens. A student picks up a pen at random, checks its colour and replaces it in the box. He repeats this process 3 times. The probability that at least one pen picked was red is:

(a) $\frac{124}{125}$

(b) $\frac{1}{125}$

(c) $\frac{61}{125}$

(d) $\frac{64}{125}$

[CBSE 2025]

Solution

(c) : P (getting red) $= \frac{3}{15} = \frac{1}{5}$

P(not getting red) $= 1 - \frac{1}{5} = \frac{4}{5}$

$\therefore$ P(at least one pen is red in three trials)

$= 1 - P$ (not getting red in three trials)

$= 1 - \left(\frac{4}{5}\right)^3 = 1 - \frac{64}{125} = \frac{125-64}{125} = \frac{61}{125}$

23. Chances that three persons A, B, and C go to the market are $30\%, 60\%$ and 50% respectively. The probability that at least one will go to the market is:

(a) $\frac{14}{10}$

(b) $\frac{43}{50}$

(c) $\frac{9}{100}$

(d) $\frac{7}{50}$

[CBSE 2025]

Solution

. (b) : Given, $P(A) = 30\% = \frac{30}{100}, P(B) = \frac{60}{100}$

and $P(C) = \frac{50}{100}$

$\therefore$ Probability that atleast one will go the market

$= 1 - P$ (none will go to the market)

$= 1 - P(\bar{A})P(\bar{B})P(\bar{C})$

$= 1 - \left(1 - \frac{30}{100}\right)\left(1 - \frac{60}{100}\right)\left(1 - \frac{50}{100}\right)$

$$= 1 - \frac{70}{100} \times \frac{40}{100} \times \frac{50}{100} = 1 - \frac{14}{100} = \frac{86}{100} = \frac{43}{50}$$

24. For the vacancy advertised in the newspaper, 3000 candidates submitted their applications. From the data it was revealed that two third of the total applicants were females and other were males. The selection for the job was done through a written test. The performance of the applicants indicates that the probability of a male getting a distinction in written test is 0.4 and that a female getting a distinction is 0.35 . Find the probability that the candidate chosen at random will have a distinction in the written test.

[CBSE 2025]

Solution

Total candidates $= 3000$

Number of females $= \frac{2}{3}(3000) = 2000$

Number of males = 1000

Let A be the event that a candidate will have a distinction in the written test.

E_1 be the event that a candidate is male.

E_2 be the event that a candidate is female.

Now, $P\left(\frac{A}{E_1}\right) = 0.4, P\left(\frac{A}{E_2}\right) = 0.35$

$\therefore P(E_1) = \frac{1000}{3000} = \frac{1}{3}, P(E_2) = \frac{2000}{3000} = \frac{2}{3}$

Now, $P(A) = P(E_1) \cdot P\left(\frac{A}{E_1}\right) + P(E_2) \cdot P\left(\frac{A}{E_2}\right)$

$$= \frac{1}{3}\left(\frac{4}{10}\right) + \frac{2}{3}\left(\frac{35}{100}\right) = \frac{4}{30} + \frac{70}{300} = \frac{4}{30} + \frac{7}{30} = \frac{11}{30}$$

25. Case Study : A shop selling electronic items sells smartphones of only three reputed companies A, B and C because chances of their manufacturing a defective smartphone are only $5\%, 4\%$ and 2% respectively. In his inventoryhehas 25% smartphones from company $A, 35\%$ smartphones from company B and 40% smartphones from company C.

A person buys smartphone from this shop.

(i) Find the probability that it was defective.

(ii) What is the probability that this defective smartphone was manufactured by company B ?

[CBSE 2025]

Solution

(i) Let E_1, E_2 and E_3 are the events that thesmartphones was manufactured by companies A, B and C, respectively.

$P(E_1) = \frac{25}{100} = 0.25, P(E_2) = \frac{35}{100} = 0.35, P(E_3) = \frac{40}{100} = 0.4$

Let A be the event that smartphones was defective.

$$P(A \mid E_1) = 5\% = 0.05$$
$$P(A \mid E_2) = 4\% = 0.04$$
$$P(A \mid E_3) = 2\% = 0.02$$

$P(A) = P(E_1) \cdot P(A \mid E_1) + P(E_2) \cdot P(A \mid E_2) + P(E_3) \cdot P(A \mid E_3)$
$= 0.25 \times 0.05 + 0.35 \times 0.04 + 0.4 \times 0.02$
$= 0.0125 + 0.014 + 0.008 = 0.0345$
$P(E_2 \mid A) = \frac{P(E_2)P(A|E_1)}{P(A)}$
$= \frac{0.35 \times 0.04}{0.0345} = \frac{0.014}{0.0345} = \frac{28}{69} = 0.41$

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